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Cohomogeneity One Manifolds of Spin(7) and G_2 Holonomy

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ABSTRACT

In this paper, we look for metrics of cohomogeneity one in $D = 8$ and $D = 7$ dimensions with Spin(7) and G_2 holonomy respectively. In $D = 8$, we first consider the case of principal orbits that are S^7, viewed as an S^3 bundle over S^4 with triaxial squashing of the S^3 fibres. This gives a more general system of first-order equations for Spin(7) holonomy than has been solved previously. Using numerical methods, we establish the existence of new non-singular asymptotically locally conical (ALC) Spin(7) metrics on line bundles over $\mathbb{CP}^3$, with a non-trivial parameter that characterises the homogeneous squashing of $\mathbb{CP}^3$. We then consider the case where the principal orbits are the Aloff-Wallach spaces $N(k, \ell) = SU(3)/U(1)$, where the integers k and ℓ characterise the embedding of $U(1)$. We find new ALC and AC metrics of Spin(7) holonomy, as solutions of the first-order equations that we obtained previously in hep-th/0102185. These include certain explicit ALC metrics for all $N(k, \ell)$, and numerical and perturbative results for ALC families with AC limits. We then study $D = 7$ metrics of G_2 holonomy, and find new explicit examples, which, however, are singular, where the principal orbits are the flag manifold $SU(3)/(U(1) \times U(1))$. We also obtain numerical results for new non-singular metrics with principal orbits that are $S^3 \times S^3$. Additional topics include a detailed and explicit discussion of the Einstein metrics on $N(k, \ell)$, and an explicit parameterisation of $SU(3)$.

Contents

1 Introduction

Metrics of special holonomy are of considerable interest both in mathematics and in physics. They are special cases of Ricci-flat metrics, whose holonomy groups are strictly smaller than the $SO(D)$ holonomy of a generic D-dimensional metric. The irreducible cases include Ricci-flat Kähler metrics in dimension $D = 2n$, with holonomy $SU(n)$, and hyper-Kähler metrics in dimension $D = 4n$, with holonomy $Sp(n)$. Two further irreducible cases arise, namely G_2 holonomy in $D = 7$, and Spin(7) holonomy in $D = 8$. It is to these latter cases, known as metrics of exceptional holonomy, that this paper is devoted.

Our focus in this paper will be on non-compact metrics of cohomogeneity one, in dimensions $D = 7$ and 8. The first complete and non-singular such examples were obtained [1], and first appeared in the physics literature in [2]. They comprised three metrics of G_2 holonomy in $D = 7$, and one of Spin(7) holonomy in $D = 8$, and all four of these metrics are asymptotically conical (AC). Specifically, the metrics in $D = 7$ are asymptotic to cones over $S^3 \times S^3$, $\mathbb{CP}^3$, and the six-dimensional flag manifold $SU(3)/(U(1) \times U(1))$, while the metric in $D = 8$ is asymptotic to a cone over S^7. In all cases the base of the cone carries an Einstein metric, albeit not the "standard" one. Topologically, the three $D = 7$ manifolds are the spin bundle of S^3, and the bundles of self-dual 2-forms over S^4 and $\mathbb{CP}^2$ respectively.

The topology of the $D = 8$ manifold is the chiral spin bundle of S^4. The homogeneous S^7 of the principal orbits can be described as a (round) S^3 bundle over S^4, with the sizes of the S^3 fibre and the S^4 base being functions of the radial variable. The specific forms of these functions ensure that the metric is complete on the chiral spin bundle of S^4, with the radius of the S^3 fibres approaching zero at short distance in such a way that one obtains the required non-singular $\mathbb{R}^4$ bundle over S^4.

Recently, further complete and non-singular non-compact 8-metrics of Spin(7) holonomy were found [3]. By contrast to the example in [1, 2], the new metrics are asymptotically locally conical (ALC), approaching the product of a circle and an AC 7-manifold locally at large distance. The 7-manifold is a cone over $\mathbb{CP}^3$. The new metrics were obtained by writing a metric ansatz with a more general parameterisation of homogeneous metrics on the S^7 principal orbits, in which the S^3 fibres over S^4 can themselves be "squashed," with the S^3 described as an S^1 bundle over S^2. This now gives functions in the metric ansatz, parameterising the sizes of the S^4, S^2 and S^1. First-order equations for these functions were derived in [3], which can be viewed as the necessary conditions for Spin(7) holonomy, and then the general solution was obtained. Besides the previous AC example of [1, 2], which of course is contained as a special case, all the new solutions are ALC. The ALC nature of the large-distance behaviour arises because the function parameterising the size of the S^1 tends to a constant at infinity. The general solution of the first-order equations has a family of non-singular metrics, with a non-trivial continuous parameter (i.e. a parameter over and above the trivial scale size). In general the manifold is again the chiral spin bundle of S^4. For general values of the parameter the solution is quite complicated, and is expressed in terms of hypergeometric functions; the manifolds were denoted by $\mathbb{B}^+$ and $\mathbb{B}^-$ in [3]. For a particular value of the parameter the solution becomes much simpler, and is expressible in terms of rational functions; this case was denoted by $\mathbb{B}_8$ in [3]. One further complete solution arises; an isolated example which is topologically $\mathbb{R}^8$, and denoted by $\mathbb{A}_8$ in [3].

In section 2, we consider a further generalisation of the ansatz for Spin(7) metrics with S^7 principal orbits, in which the S^3 fibres over the S^4 base have "triaxial" homogeneous distortions, implying that there will now be a total of four functions parameterising the various radii. We obtain first-order equations that imply Spin(7) holonomy, and then we discuss the possible solutions. The previous examples in [1, 2] and [3] of course arise as special cases. Although we have not been able to obtain more general solutions analytically, we have carried out an extensive numerical analysis of the equations. We find clear evidence for the existence of non-singular triaxial solutions, in which there is a minimal $\mathbb{CP}^3$ surface (a bolt) at short distance, with an ALC behaviour at infinity. There is a non-trivial one-parameter family of such regular solutions where the parameter can be thought of as characterising the "squashing" of the minimal $\mathbb{CP}^3$, viewed as an S^2 bundle over S^4. If we denote the ratio of the radius of S^2 over the radius of $\mathbb{CP}^2$ by λ, then we find non-singular metrics for $\lambda^2 \le 4$. The special case $\lambda^2 = 4$ corresponds to the "round" Fubini-Study metric on $\mathbb{CP}^3$, and in this case the 8-metric is nothing but the complex line bundle over $\mathbb{CP}^3$ contained in [4, 5], which has the smaller holonomy $SU(4)$. When $\lambda^2 < 4$, the new metrics exhibit a behaviour reminiscent of the Atiyah-Hitchin [6] hyper-Kähler 4-metric, with the three radial functions a_i on S^3 going from $a_1^2 = 0$, $a_2^2 = a_3^2 =$constant at the bolt, to $a_3 =$constant, $a_1^2 \sim a_2^2 \sim r^2$ at large radius r. We denote these new 8-manifolds of Spin(7) holonomy by $\mathbb{C}_8$.

In section 3, we examine 8-metrics of Spin(7) holonomy where the principal orbits are the Aloff-Wallach homogeneous spaces $N(k,\ell)$, which are $SU(3)/U(1)$ with the integers k and ℓ specifying the embedding of the $U(1)$ in $SU(3)$. We begin, in section 3.1, by reviewing some of the relevant properties of the Aloff-Wallach spaces themselves. In particular, we present a more explicit demonstration than has previously appeared in the literature of the fact that for generic k and ℓ, each $N(k,\ell)$ admits two inequivalent Einstein metrics. (The existence of an Einstein metric for each $N(k,\ell)$ was proven in [7]; an explicit expression for one such metric on $N(k,\ell)$ was given in [8], and the proof that each generic $N(k,\ell)$ admits two Einstein metrics was given in [9].) We also give an alternative proof of a result following from [8, 9] that every homogeneous Einstein metric has weak G_2 holonomy. In section 3.2, we give a discussion of the global structures of the $N(k,\ell)$ spaces, focusing in particular on the question of when a given such space admits a description as an S^3 (as opposed to lens-space) bundle over $\mathbb{CP}^2$. This is important in what follows in section 4, where we discuss details of 8-metrics of Spin(7) holonomy with $N(k,\ell)$ principal orbits. The first-order equations for Spin(7) holonomy for this class of metrics were obtained in [10]. In order to have a non-singular such metric on an $\mathbb{R}^4$ bundle over $\mathbb{CP}^2$, it is crucial that the collapsing fibres at short distance should be S^3 and not a lens space. We obtain an explicit analytical local solution, which is ALC, for each choice of $N(k,\ell)$ principal orbit. We also give a discussion of numerical solutions, which indicate the existence of complete examples with a non-trivial parameter, and which include metrics that are asymptotically-conical in a particular limit.

In section 5, we turn to a consideration of more general 7-metrics of G_2 holonomy. We begin in section 5.1 by studying 7-metrics of G_2 holonomy on the $\mathbb{R}^3$ bundle of self-dual 2-forms over $\mathbb{CP}^2$. These generalise the AC example on this topology in [1, 2], whose principal orbits are the flag manifold $SU(3)/(U(1)\times U(1))$, with two size parameters as metric functions. The more general ansatz that we consider here has principal orbits of the same topology, but with three, instead of two, sizes as metric functions. Interestingly, the first-order equations that follow from requiring G_2 holonomy turn out to be the same as those that arise in four dimensions, for a set of Bianchi IX hyper-Kähler metrics. In that case the equations was solved completely in [11], and so we are able to use the same procedure here. As in the four-dimensional case, we find here that the general solution gives irregular metrics, with regularity attained only if two of the metric functions are equal, which reduces the system to the already-known one in [1, 2].

In section 5.2, we briefly consider the possibility of more general 7-metrics of G_2 holonomy where the principal orbits are $\mathbb{CP}^3$. Although we end up concluding that no possibilities of greater generality than those considered previously in [1, 2] arise, we do nevertheless obtain as a by-product a more elegant formulation of the already-known metrics.

In section 5.3, we study a general system of equations for metrics of G_2 holonomy on the spin bundle of S^3. Here, the principal orbits have the topology of $S^3\times S^3$. A rather general ansatz with six functions parameterising sizes in a family of squashed $S^3\times S^3$ metrics was studied in [12, 13], where first-order equations implying G_2 holonomy were derived. We find by means of a numerical analysis that the only non-singular solutions occur when two pairs of metric functions are set equal, leading to a truncation to a four-function system that was discussed in [13], where an isolated non-singular ALC solution was obtained explicitly. Perturbative arguments in [13] suggested that a more general family of non-singular ALC solutions, with a non-trivial parameter, should exist. These would be analogous to the 1-parameter family of ALC Spin(7) metrics found in [3]. Using a numerical analysis, we also find evidence for the existence of such a 1-parameter family of non-singular solutions. We denote these by $\mathbb{B}_7^+$ and $\mathbb{B}_7^-$, with the isolated example found in [13] being denoted by $\mathbb{B}_8$.

After conclusions, we include a number of appendices. Appendix A contains a detailed discussion of the parameterisation of $SU(3)$ in terms of generalised Euler angles, which is useful in our discussion of the global structure of the Aloff-Wallach $N(k,\ell)$ spaces.

Recent applications of Ricci-flat manifolds with special holonomy in string and M-theory can be found in [14-45].

2 New Spin(7) metrics with triaxial S^3 bundle over S^4

In [3], new complete non-singular Spin(7) metrics on the chiral spin bundle of S^4, and on $\mathbb{R}^8$, were constructed. These metrics have cohomogeneity one, with principal orbits that are S^7, with a transitively-acting $SO(5)\times U(1)$ isometry, and they are asymptotically locally conical (ALC). They were obtained by generalising the original ansatz used in the AC example of [1, 2], by describing S^7 as an S^3 bundle over S^4, with radial functions in the metric parameterising the size of the S^4 base, and the sizes of S^2 and the $U(1)$ fibres in a description of S^3 as the Hopf bundle over S^2. First-order equations coming from a superpotential were then constructed, and the general solution was obtained. A 1-parameter family of non-singular solutions on the chiral spin bundle over S^4 was obtained; these were denoted by $\mathbb{B}_8^+$, $\mathbb{B}_8^-$ and $\mathbb{B}_8$ in [3]. It should be emphasised that the parameter in these solutions is non-trivial, and not merely a scale size. The general solution also includes an isolated non-singular Spin(7) metric on $\mathbb{R}^8$; this was denoted by $\mathbb{A}_8$ in [3]. The local form of the metric in this example is in fact the same as the metric on $\mathbb{B}_8$ in the 1-parameter family on the chiral spin bundle of S^4, but with the range of the radial coordinate chosen differently.

In this section, we shall generalise the construction in [3], by introducing a fourth radial function in the cohomogeneity one metrics, so that the principal orbits are now S^7 described as a bundle of triaxially-squashed 3-spheres over S^4. After calculating the curvature, we find that the potential in a Lagrangian description of the Ricci-flat conditions can be derived from a superpotential, and hence we obtain a system of first-order equations for the four metric functions. These are equivalent to the integrability conditions for Spin(7) holonomy. In fact the equations that we obtain have also been found recently by Hitchin [47], using a rather different method.

2.1 Ansatz and first-order equations

We begin by introducing left-invariant 1-forms L_{AB} for the group manifold $SO(5)$. These satisfy $L_{AB} = -L_{BA}$, and

$$dL_{AB} = L_{AC} \wedge L_{CB}\,. \tag{1}$$

The 7-sphere is then given by the coset $SO(5)/SU(2)_L$, where we take the obvious $SO(4)$ subgroup of $SO(5)$, and write it (locally) as $SU(2)_L \times SU(2)_R$.

If we take the indices A and B in L_{AB} to range over the values $0 \le A \le 4$, and split them as $A = (a, 4)$, with $0 \le a \le 3$, then the $SO(4)$ subgroup is given by L_{ab}. This is decomposed as $SU(2)_L \times SU(2)_R$, with the two sets of $SU(2)$ 1-forms given by the self-dual and anti-self-dual combinations:

$$R_i = \tfrac{1}{2}(L_{0i} + \tfrac{1}{2}\epsilon_{ijk}\, L_{jk})\,, \qquad L_i = \tfrac{1}{2}(L_{0i} - \tfrac{1}{2}\epsilon_{ijk}\, L_{jk})\,, \tag{2}$$

where $1 \le i \le 3$. Thus the seven 1-forms in the S^7 coset will be

$$P_a \equiv L_{a4}\,, \qquad R_1\,, \qquad R_2\,, \qquad R_3\,. \tag{3}$$

It is straightforward to establish that

$$\begin{aligned}
dP_0 &= (R_1 + L_1) \wedge P_1 + (R_2 + L_2) \wedge P_2 + (R_3 + L_3) \wedge P_3\,,\\
dP_1 &= -(R_1 + L_1) \wedge P_0 - (R_2 - L_2) \wedge P_3 + (R_3 - L_3) \wedge P_2\,,\\
dP_2 &= (R_1 - L_1) \wedge P_3 - (R_2 + L_2) \wedge P_0 - (R_3 - L_3) \wedge P_1\,,\\
dP_3 &= -(R_1 - L_1) \wedge P_2 + (R_2 - L_2) \wedge P_1 - (R_3 + L_3) \wedge P_0\,,\\
dR_1 &= -2R_2 \wedge R_3 - \tfrac{1}{2}(P_0 \wedge P_1 + P_2 \wedge P_3)\,,\\
dR_2 &= -2R_3 \wedge R_1 - \tfrac{1}{2}(P_0 \wedge P_2 + P_3 \wedge P_1)\,,\\
dR_3 &= -2R_1 \wedge R_2 - \tfrac{1}{2}(P_0 \wedge P_3 + P_1 \wedge P_2)\,.
\end{aligned} \tag{4}$$

We are now in a position to write an ansatz for the more general metrics of Spin(7) holonomy on the $\mathbb{R}^4$ bundle over S^4,

$$ds_8^2 = dt^2 + a_i^2\, R_i^2 + b^2\, P_a^2\,. \tag{5}$$

From this, we find after mechanical calculations using (4) that the conditions for Ricci-flatness can be derived from the Lagrangian $L = T - V$, together with the constraint $T + V = 0$, where

$$\begin{aligned} T &= 2\alpha_1'\,\alpha_2' + 2\alpha_2'\,\alpha_3' + 2\alpha_1'\,\alpha_3' + 8(\alpha_1' + \alpha_2' + \alpha_3')\,\alpha_4' + 12{\alpha_4'}^2\,,\\ V &= \tfrac14 a_1^2\,a_2^2\,a_3^2\,b^4\,(a_1^2 + a_2^2 + a_3^2) + 2b^8\,(a_1^4 + a_2^4 + a_3^4 - 2a_1^2\,a_2^2 - 2a_2^2\,a_3^2 - 2a_1^2\,a_3^2)\\ &\quad -12a_1^2\,a_2^2\,a_3^2\,b^6\,, \end{aligned} \tag{6}$$

where $a_i = e^{\alpha_i}$, $b = e^{\alpha_4}$, and a prime denotes a derivative with respect to η, defined by $dt = a_1^2\,a_2^2\,a_3^2\,b^8\,d\eta$. Reading off the DeWitt metric g_{ij} from the kinetic energy $T = \frac12 g_{ij}\,\alpha^{i'}\,\alpha^{j'}$, we find that the potential V can be written in terms of a superpotential W, as $V = -\frac12 g^{ij}\,(\partial W/\partial\alpha^i)\,(\partial W/\partial\alpha^j)$, where

$$W = a_1\,a_2\,a_3\,(a_1 + a_2 + a_3)\,b^2 - 2b^4\,(a_1^4 + a_2^4 + a_3^4 - 2a_1\,a_2 - 2a_2\,a_3 - 2a_3\,a_1)\,. \tag{7}$$

This leads to the first-order equations $\alpha^{i'} = g^{ij}\,\partial W/\partial\alpha^j$, which gives

$$\begin{aligned} \dot a_1 &= \frac{a_1^2 - (a_2 - a_3)^2}{a_2\,a_3} - \frac{a_1^2}{2b^2}\,,\\ \dot a_2 &= \frac{a_2^2 - (a_3 - a_1)^2}{a_3\,a_1} - \frac{a_2^2}{2b^2}\,,\\ \dot a_3 &= \frac{a_3^2 - (a_1 - a_2)^2}{a_1\,a_2} - \frac{a_3^2}{2b^2}\,,\\ \dot b &= \frac{a_1 + a_2 + a_3}{4b}\,, \end{aligned} \tag{8}$$

where a dot denotes a derivative with respect to the original radial variable t appearing in the ansatz (5). It is straightforward to see that these are in fact the integrability conditions for Spin(7) holonomy.[1]

2.2 Some properties of the equations

2.2.1 Truncations to simpler systems

First, note that if we drop the terms associated with b, we get precisely the first-order system that arises for triaxial Bianchi IX metrics in $D = 4$ [46] that admits the Atiyah-Hitchin

[1] These equations were also obtained recently by N. Hitchin, using a rather different construction [47].

metric [6] as a solution. This corresponds to a limit in which the radius of the S^4 goes to infinity, so that we effectively recover the equations for Atiyah-Hitchin times flat $\mathbb{R}^4$. Some properties of the Atiyah-Hitchin solutions are reviewed in Appendix B.

If instead we set an two of the a_i equal, say $a_2 = a_3$, and make the redefinitions $a_2 = a_3 \longrightarrow 2a$, $a_1 \longrightarrow 2b$, $b \longrightarrow c$, we get precisely the first-order system of our previous paper [3] on the new Spin(7) manifolds $\mathbb{A}_8$, $\mathbb{B}_8$ and $\mathbb{B}_8^{\pm}$, namely

$$\dot{a} = 1 - \frac{b}{2a} - \frac{a^2}{c^2}\,, \qquad \dot{b} = \frac{b^2}{2a^2} - \frac{b^2}{c^2}\,, \qquad \dot{c} = \frac{a}{c} + \frac{b}{2c}\,. \tag{9}$$

This system was solved completely in [3].

A third specialisation is to set $a_2 = -a_3$. It can be seen from (8) that this will be consistent provided that we also impose $a_2 = 2b$. We then have the metric ansatz

$$ds_8^2 = dt^2 + a_1^2\, R_1^2 + a_2^2\, (R_2^2 + R_3^2 + \tfrac14 P_a^2)\,, \tag{10}$$

and the truncated first-order equations are the equations

$$\dot{a}_1 = 4 - \frac{3a_1^2}{a_2^2}\,, \qquad \dot{a}_2 = \frac{a_1}{a_2}\,. \tag{11}$$

It is straightforward to solve these, to give

$$ds_8^2 = \Big(1 - \frac{\ell^8}{r^8}\Big)^{-1}\, dr^2 + r^2\, \Big(1 - \frac{\ell^8}{r^8}\Big)\, R_1^2 + r^2\, (R_2^2 + R_3^2 + \tfrac14 P_a^2)\,. \tag{12}$$

This is in fact precisely the Ricci-flat Kähler metric with $SU(4) \equiv$ Spin(6) holonomy, on an $\mathbb{R}^2$ bundle over $\mathbb{CP}^3$. The complete metric is asymptotic to the cone over $S^7/\mathbb{Z}_4$. The 6-metric with a_2^2 prefactor is in fact precisely the Fubini-Study metric on $\mathbb{CP}^3$.

2.2.2 Some observations about the Spin(7) system

Let us define w_i variables as in the Atiyah-Hitchin case [6], namely

$$w_1 = a_2\, a_3\,, \qquad w_2 = a_3\, a_1\,, \qquad w_3 = a_1\, a_2\,, \tag{13}$$

and a radial variable η by $dt = a_1\, a_2\, a_3\, d\eta$ (exactly as for Atiyah-Hitchin). Also, define

$$\beta \equiv b^2\,. \tag{14}$$

Then the first-order equations (8) become

$$\begin{aligned}
\frac{d(w_1+w_2)}{d\eta} &= 4w_1\, w_2 - \frac{1}{2\beta}\, [w_1\, w_2\, (w_1+w_2) + w_3\, (w_1^2 + w_2^2)]\,,\\
\frac{d(w_2+w_3)}{d\eta} &= 4w_2\, w_3 - \frac{1}{2\beta}\, [w_2\, w_3\, (w_2+w_3) + w_1\, (w_2^2 + w_3^2)]\,,\\
\frac{d(w_3+w_1)}{d\eta} &= 4w_3\, w_1 - \frac{1}{2\beta}\, [w_3\, w_1\, (w_3+w_1) + w_2\, (w_3^2 + w_1^2)]\,,\\
\frac{d\beta}{d\eta} &= \tfrac12 (w_1\, w_2 + w_2\, w_3 + w_3\, w_1)\,.
\end{aligned} \tag{15}$$

This set of equations can be reduced to a single highly non-linear second-order equation. To do this, we first make the field redefinitions

$$X = w_3 - w_1\,, \qquad Y = w_2 - w_1\,, \qquad Z = w_2 + w_3\,. \tag{16}$$

We can then derive the following simple equations

$$\frac{d}{d\eta}\log(\frac{X}{Y}) = 4(Y - X)\,, \qquad \frac{d}{d\eta}\log(X\,Y\,\beta^2) = 4Z\,, \tag{17}$$

together with two more complicated equations for $\dot{Z}$ and $\dot{\beta}$. In terms of a new radial variable τ, defined by $d\tau = Z\,d\eta$, we therefore have

$$\beta^2\,X\,Y\,Z^{-2} = c_0\,e^{4\tau}\,, \tag{18}$$

where c_0 is a constant of integration. After the further redefinitions

$$U \equiv \frac{X - Y}{Z}\,, \qquad V \equiv \frac{X}{Y}\,, \qquad \tilde{\beta} = \beta\,Z^{-1}\,, \tag{19}$$

and the introduction of another radial variable z defined by $dz = 8U\,V\,(V-1)^{-2}\,d\tau$, we find that V is given by

$$V = \frac{z+1}{z-1}\,, \tag{20}$$

and then the remaining two equations, for U and $\tilde{\beta}$, become

$$\begin{aligned} U' &= \frac{U^2 - 2z\,U + 1}{2(z^2-1)} - \frac{(U^2-1)(z\,U-1)}{16(z^2-1)\,\tilde{\beta}}\,, \\ \tilde{\beta}' &= \frac{\tilde{\beta}\,(U^2-1)}{2(z^2-1)\,U} - \frac{z\,U^3 + U^2 + 3z\,U - 5}{16(z^2-1)\,U}\,, \end{aligned} \tag{21}$$

where a prime means d/dz. One can solve algebraically for $\tilde{\beta}$ in the U' equation, and substitute it back into the $\tilde{\beta}'$ equation, thereby obtaining a second-order non-linear equation for U:

$$\begin{aligned} &2(z^2-1)^2\Big(U\,(U^2-1)\,(z\,U-1)\,U'' - (2z\,U^3 - 3U^2 - 4z\,U + 5)\,U'^2\Big) \\ &\quad +(z^2-1)\Big(2(z\,U-1)^2\,(U^2+5) - (U^2-1)^2\Big)\,U' \\ &\qquad +2(z\,U-1)^2\,(z\,U^3 - 3U^2 + 3z\,U - 1) = 0\,. \end{aligned} \tag{22}$$

It is not clear how to solve this equation in general. Here, we just remark that two special solutions are $U = 2\,(z+1)^{-1}$ and $U = 2(z-1)^{-1}$. In fact both of these correspond to the previously-known solution (12) on the complex line bundle over $\mathbb{CP}^3$. For example, for $U = 2(z+1)^{-1}$, after defining a radial coordinate r by $r^8 = \frac{1}{16}(z+3)^2\,(z+1)^{-1}$, one obtains (12) with $\ell^8 = \frac{1}{2}$. (The $\mathbb{A}_8$ and $\mathbb{B}_8$ metrics found in [3] are also special solutions of the more general triaxial system we are studying here, but these all correspond to a degeneration of the parameterisation in this section, with $Y = 0$ or $X = Y$.)

2.3 Numerical analysis

Since we have not been able to solve the first-order equations (8) explicitly, we now turn to a numerical analysis of the equations. In the case of non-singular manifolds, the metrics are defined on $\mathbb{R}^+ \times G/H$, completed by the addition of a degenerate orbit G/K at short distance, where K contains H. The possible cases are $G/K = SO(5)/U(2) = \mathbb{CP}^3$, $G/K = SO(5)/SO(4) = S^4$, or $G/K = SO(5)/SO(5) = \mathbb{1}$, corresponding to a $\mathbb{CP}^3$ or S^4 bolt, or a NUT, respectively.

Our technique consists of first performing an analytic Taylor expansion of the solution in the neighbourhood of the degenerate orbit (i.e. in the neighbourhood of the NUT or bolt at short distance). When making this expansion, we impose the necessary boundary conditions to ensure that the metric can be regular there (in an appropriate coordinate system). At this stage we are left with a number of undetermined coefficients in the Taylor expansions, and these represent the free parameters in the general solution that is non-singular near the NUT or bolt. We then use these Taylor expansions in order to determine initial conditions a short distance away from the NUT or bolt, and then we evolve these data to large distance in a numerical integration of the first-order equations (8). In particular, we can study the evolution of the solution as a function of the free parameters, and determine the conditions under which the solution is non-singular.

We find that the non-singular solutions are those where the metric at large distance approaches either a cone over S^7/Z_4 (the AC line bundle over $\mathbb{CP}^3$ case) or S^7 (the AC chiral spin bundle over S^4), or else a circle splits off and approaches a constant radius, while the other directions in the principal orbits grow linearly so that the metric approaches S^1 times a cone over $\mathbb{CP}^3$ locally. These are the ALC cases. In fact, we find that generically the non-singular solutions are ALC, with the AC behaviour arising only as a limiting case.

In cases where the initial conditions do not lead to an ALC or AC structure at infinity, we find from the numerical analysis that a singularity arises in the metric functions. In other words, the set of non-singular metrics corresponds to those cases where the choice of initial conditions leads to an ALC or AC behaviour at infinity.

2.3.1 Numerical analysis for $\mathbb{CP}^3$ bolts: New Spin(7) metrics $\mathbb{C}_8$

We can study the solution space for regular metrics in this case as follows. First, we seek a solution in the form of a Taylor series in t, for small t, that exhibits the required short-distance behaviour. In the present context, where we are looking for solutions in which the metric collapses to a $\mathbb{CP}^3$ bolt at $t=0$, we make an expansion

$$a_i(t) = \sum_{n\geq 0} x_i(n)\, t^n\,, \qquad b(t) = \sum_{n\geq 0} y(n)\, t^n\,, \tag{23}$$

where $x_1(0)=0$, implying that a_1 vanishes at $t=0$. Thus R_2 and R_3 describe the directions on an S^2 bundle over the S^4 that is described by the P_a.

We find that the general Taylor expansion of this form has 2 free parameters. These can be taken to be $y(0)$, specifying the radius of the S^4 base, and $x_2(0)$ (which is equal to $-x_3(0)$) specifying the radius of the S^2 fibres, on the $\mathbb{CP}^3$ bolt at $t=0$. One of these two parameters is trivial, corresponding merely to a choice of overall scale, and so without loss of generality we may take $y(0)=1$. For convenience, we shall define $x_2(0)\equiv\lambda$, and so this corresponds to a non-trivial adjustable parameter in the solutions that are regular near the $\mathbb{CP}^3$ bolt. Thus the metric restricted to the bolt is

$$ds_6^2 = \lambda^2\,(R_2^2+R_3^2) + P_a^2\,. \tag{24}$$

Note that this family of homogeneous metrics on $\mathbb{CP}^3$ reduces to the standard Fubini-Study metric if $\lambda^2=4$.

To the first few orders in t, we find that the Taylor-expanded solution to (8) is given by

$$\begin{aligned}
a_1 &= 4t + \frac{(\lambda^4-40\lambda^2-48)}{12\lambda^2}\, t^3 + \cdots\,,\\
a_2 &= \lambda + (1-\tfrac14\lambda^2)\, t + \frac{(3\lambda^4-8\lambda^2+48)}{32\lambda}\, t^2 + \cdots\,,\\
a_3 &= -\lambda + (1-\tfrac14\lambda^2)\, t - \frac{(3\lambda^4-8\lambda^2+48)}{32\lambda}\, t^2 + \cdots\,,\\
b &= 1 + \tfrac{1}{16}(12-\lambda^2)\, t^2 + \cdots\,.
\end{aligned} \tag{25}$$

Using the Taylor expansions to set initial data at some small positive value of t, we now evolve the equations (8) forward to large t numerically.[2] We find that the solution with the above small-t behaviour is regular provided that the non-trivial parameter λ is chosen to that

$$\lambda^2 \leq 4\,. \tag{26}$$

The case $\lambda^2=4$ corresponds precisely to the situation we arrived at in (10). Namely, setting $\lambda^2=4$ in the $\mathbb{CP}^3$ bolt metric (24), we get precisely the Fubini-Study metric on $\mathbb{CP}^3$. In fact the solution when $\lambda^2=4$ is nothing but the Ricci-flat Kähler metric given by (10).

[2] In order to set the initial data accurately at a sufficient distance away from the singular point of the equations at $t=0$, we typically perform the Taylor expansions to tenth order in t.

This is the limiting AC case that we alluded to above. It has an "accidental" decrease in its holonomy group from the Spin(7) of the generic solution of (8) to $SU(4) \equiv$ Spin(6), with a consequence increase from 1 to 2 parallel spinors.

When $\lambda^2 < 4$, we get new non-singular solutions, which we shall denote by $\mathbb{C}_8$. From the numerical analysis, we find that now, the metric function a_3 tends to a constant value at large distance, while all the others grow linearly. Thus for $\lambda^2 < 4$ the solution is ALC. The case where λ^2 becomes zero is a Gromov-Hausdorff limit in which the metric becomes the product $M_4 \times \mathbb{R}^4$, where M_4 is the Atiyah-Hitchin metric. The λ modulus space of the new solutions is depicted in Figure 1 below (we assume, without loss of generality, that λ is non-negative, so that regular solutions occur for $\lambda \leq 2$).

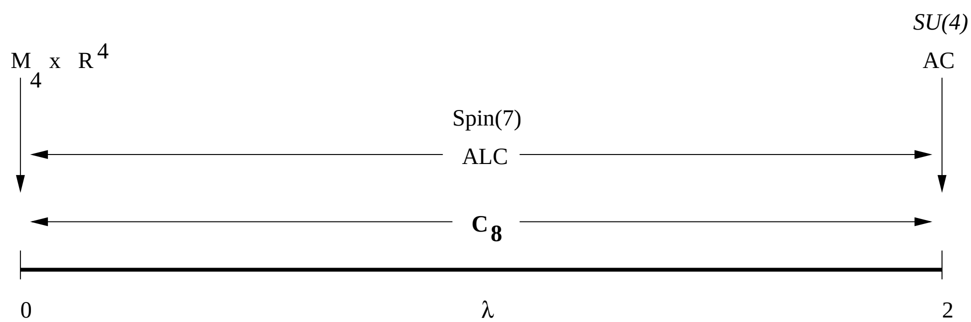

Figure 1: The new non-singular Spin(7) metrics $\mathbb{C}_8$ as a function of λ

It should be noted that the new solutions exhibit the same "slump" phenomenon that was encountered in the Atiyah-Hitchin metric. Thus, at small distance it is the a_1 direction on S^3 that is singled out, with $a_1 = 0$ while $a_2 = a_3$ =constant on the bolt. By contrast, at large distance it is a_3 that is singled out (by tending to a constant), while a_1 and a_2 become equal asymptotically. A sketch of the typical behaviour of the metric functions a_i and b is given in Figure 2 below.

It is worth remarking that the similarity of the asymptotic behaviour of the new $\mathbb{C}_8$ metrics and the Atiyah-Hitchin metric may have some interesting physical significance. Like Atiyah-Hitchin, the $\mathbb{C}_8$ metrics will have "negative mass," as measured from infinity. Just as the product of the Atiyah-Hitchin metric and seven-dimensional Minkowski spacetime describes an orientifold plane in M-theory, so too here we can expect that the $\mathbb{C}_8$ metrics will have an associated interpretation in terms of orientifolds.

The new non-singular Spin(7) metrics $\mathbb{C}_8$ have the same topology as the $\lambda = 2$ example. Thus they are line bundles over $\mathbb{CP}^3$ (specifically, the fourth power of the Hopf bundle).

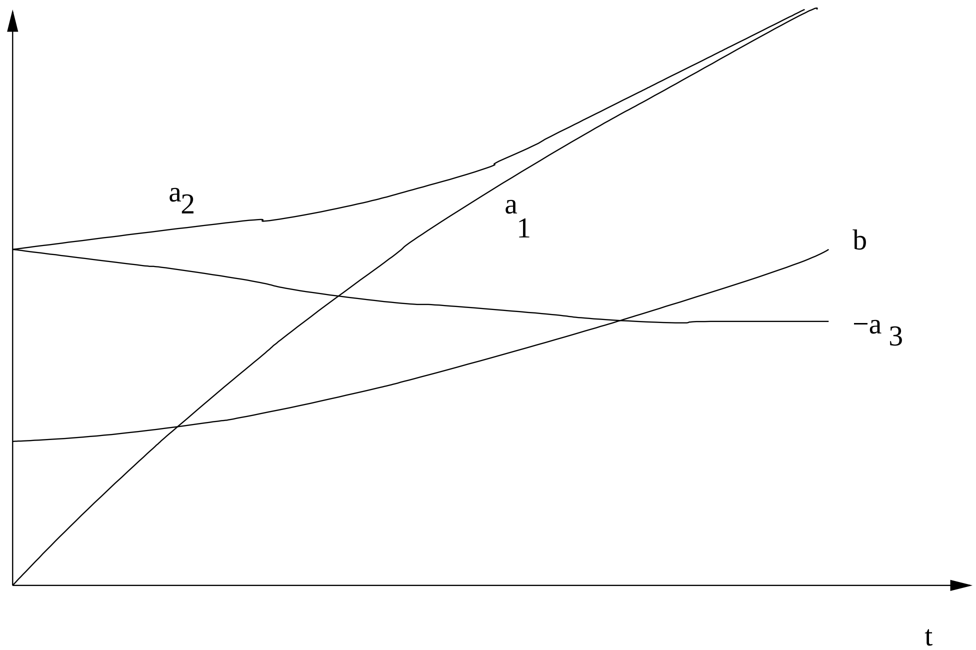

Figure 2: The metric functions for a typical non-singular ALC solution

2.3.2 Numerical analysis for S^4 bolts: The $\mathbb{B}_8$, $\mathbb{B}_8^+$ and $\mathbb{B}_8^-$ examples recovered

This case can be studied by starting from the small-distance expansion (23), and now taking $x_1(0) = x_2(0) = x_3(0) = 0$. We then find that the general such solution is characterised by three parameters, which we shall relabel as q_1, q_2 and q_3. The first few orders give

$$\begin{aligned} a_1 &= t - q_1\, t^2 \ + \cdots\,, \\ a_2 &= t - q_2\, t^2 \ + \cdots\,, \\ a_3 &= t - q_3\, t^2 \ + \cdots\,, \\ b &= b_0 + \frac{3}{8b_0}\, t^2 + \cdots\,, \end{aligned} \tag{27}$$

where $b_0 = 2^{-1/2}\,\sqrt{q_1+q_2+q_3}$. Note that we must have $q_1 + q_2 + q_3 > 0$.

From the numerical solutions we find that regularity requires that two of the q_i be equal, leading, in turn, to the equality of the corresponding pair of functions a_i. Thus all the regular solutions here reduce to ones that we have already found in [3]. It is, nevertheless, of interest to see how they relate to the previous results in [3].

Let us, without loss of generality, choose $q_2 = q_3$. This will be understood to be the case in everything that follows. The regular solutions can then be summarised as follows. In all cases we must therefore have $q_1 + 2q_2 > 0$. Regularity also turns out to imply $q_1 > 0$, and $q_2 \le q_1$. The cases are as follows:

$$\begin{aligned}
-\tfrac{1}{2}q_1 < q_2 < 0: &\qquad \text{The } \mathbb{B}_8^- \text{ metrics}\,,\\
q_2 = 0: &\qquad \text{The } \mathbb{B}_8 \text{ metric}\,,\\
0 < q_2 < q_1: &\qquad \text{The } \mathbb{B}_8^+ \text{ metrics}\,,\\
q_1 = q_2: &\qquad \text{The original AC Spin(7) metric of [1, 2]}\,.
\end{aligned} \tag{28}$$

Note that when q_2/q_1 becomes equal to $-\frac{1}{2}$, we have a Gromov-Hausdorff limit to the product $M_7 \times S^1$, where M_7 is the original AC G_2 metric [1, 2] on the $\mathbb{R}^3$ bundle over S^4. The Spin(7) metrics are depicted in Figure 3 below.

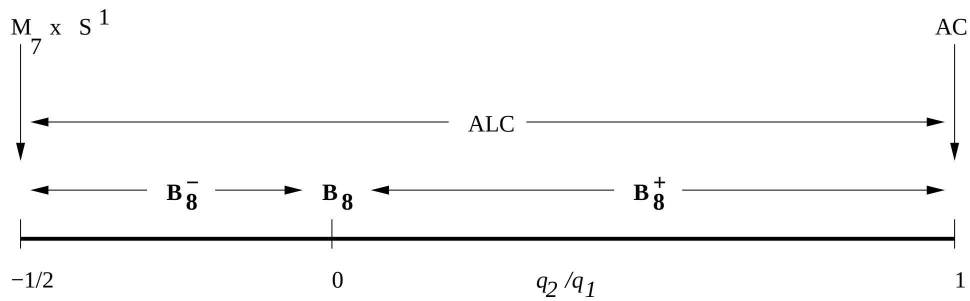

Figure 3: The non-singular Spin(7) metrics $\mathbb{B}_8$ and $\mathbb{B}_8^\pm$ as a function of q_2/q_1

2.3.3 Analysis for NUTs: The $\mathbb{A}_8$ example recovered

The short-distance expansion for the NUT case corresponds to using (23) with $x_1(0) = x_2(0) = x_3(0) = y(0) = 0$. On finds that there are only two possible solutions at short distance. One of these is

$$a_1 = a_2 = a_3 = \tfrac{3}{5}t\,, \qquad b = \frac{3}{2\sqrt{5}}\,t\,, \tag{29}$$

which is an exact solution corresponding to the cone over the squashed Einstein S^7. It is singular at the apex. The other solution has the following expansion:

$$\begin{aligned}
a_1 &= -t + q\,t^3 - \tfrac{3}{2}q^2\,t^5 + \cdots\,,\\
a_2 &= a_3 = t\,,\\
b &= \tfrac{1}{2}t + \tfrac{1}{8}q\,t^3 - \tfrac{9}{64}\,q^2\,t^5 + \cdots\,.
\end{aligned} \tag{30}$$

The constant q just corresponds to a trivial scale parameter here. This is in fact precisely the previously known solution $\mathbb{A}_8$.

In summary, we get new regular Spin(7) metrics with squashed $\mathbb{CP}^3$ bolts, with a non-trivial parameter $\lambda^2 < 4$ characterising the radius of the S^2 fibres relative to the S^4 base in the $\mathbb{CP}^3$ bolt. If $\lambda^2 = 4$, we recover the previously-known case of a complex line bundle over the Fubini-Study metric on $\mathbb{CP}^3$, with $SU(4)$ holonomy, included in the cases considered in [4, 5]. There are no new regular solutions with S^4 bolts or for NUTS, since regularity in these cases forces two of the S^3 directions to be equal, thus reducing the systems to ones already solved in [3].

3 Spin(7) metrics with $SU(3)/U(1)$ principal orbits

In this section, we shall study the solutions of the first-order equations for Spin(7) holonomy for metrics of cohomogeneity one whose principal orbits are the Aloff-Wallach spaces $N(k,\ell)$, which are cosets $SU(3)/U(1)$ where the integers k and ℓ specify the embedding of the $U(1)$. All the necessary results for the first-order Spin(7) equations were derived in [10], and here we shall also follow the notation established in that paper. We begin our discussion with a study of the Aloff-Wallach spaces themselves, and in particular the conditions under which they admit Einstein metrics and metrics of weak G_2 holonomy.

3.1 The principal orbits: Aloff-Wallach spaces

The coset spaces $SU(2)/U(1)$ are characterised by two integers, k and ℓ, which specify the embedding of the $U(1)$ in $SU(3)$. Specifically, if we represent $SU(3)$ by $3\times$ special unitary matrices then the $U(1)$ subgroup can taken to be matrices of the form

$$h = \begin{pmatrix} e^{\mathrm{i}k\,\theta} & 0 & 0 \\ 0 & e^{\mathrm{i}\,\ell\,\theta} & 0 \\ 0 & 0 & e^{-\mathrm{i}\,(k+\ell)\,\theta} \end{pmatrix}. \tag{31}$$

The coset spaces are simply-connected when k and ℓ are relatively prime, and these are denoted by $N(k,\ell)$. Clearly the spaces $N(k,\ell)$, $N(\ell,k)$ and $N(k,-k-\ell)$ are topologically identical, and in fact there is an S_3 permutation symmetry generated by these two Z_2 operations. It was shown in in [48] that all the $N(k,\ell)$ admit metrics of positive sectional curvature. Then, in [7], an existence proof for an Einstein metric on each $N(k,\ell)$ was given, and a more explicit expression was found in [8]. Subsequently, it was shown in [9] that each $N(k,\ell)$ in fact admits *two* inequivalent Einstein metrics (except when $(k,\ell) = (0,1)$ or the S_3-related values $(1,0)$ or $(1,-1)$, when there is only one). Furthermore, it was proved that each such metric admits a Killing spinor (except for one of the Einstein metrics with $k = \ell$, which admits 3 Killing spinors). The special case $k = \ell$ can be viewed as an $SO(3)$ bundle over $\mathbb{CP}^2$, and the existence of the second Einstein metric in this case had already been demonstrated in [49].

3.1.1 Einstein metrics on $N(k,\ell)$ from first-order equations

Here, we present a summary of the construction of the Aloff-Wallach spaces $N(k,\ell)$, and we give more explicit expressions for the Einstein conditions than have been presented previously. These will be useful when we study cohomogeneity one Ricci-flat metrics with $N(k,\ell)$ principal orbits in subsequent sections. A system of first-order equations following from requiring Spin(7) holonomy for such metrics was derived in [10], and we can first make use of these in order to obtain equations for Einstein metrics on $N(k,\ell)$. For further details of the construction described below, see [10].

Defining left-invariant 1-forms ${L_A}^B$ for $SU(3)$, where $A = 1,2,3$, ${L_A}^A = 0$, $({L_A}^B)^\dagger = {L_B}^A$ and $d{L_A}^B = \mathrm{i}\, {L_A}^C \wedge {L_C}^B$, we introduce the combinations

$$\begin{aligned}
&\sigma \equiv {L_1}^3\,, \qquad \Sigma \equiv {L_2}^3\,, \qquad \nu \equiv {L_1}^2\,,\\
&\lambda \equiv \sqrt{2}\,\cos\tilde\delta\, {L_1}^1 + \sqrt{2}\,\sin\tilde\delta\, {L_2}^2\,,\\
&Q \equiv -\sqrt{2}\,\sin\tilde\delta\, {L_1}^1 + \sqrt{2}\,\cos\tilde\delta\, {L_2}^2\,,
\end{aligned} \tag{32}$$

where Q is taken to be the $U(1)$ generator lying outside the $SU(3)/U(1)$ coset. It is evident by comparing with (31) that we have

$$\frac{k}{\ell} = -\tan\tilde\delta\,, \tag{33}$$

and so $\tilde\delta$ is restricted to an infinite discrete set of values. Later, it will be convenient to write

$$\cos\tilde\delta = -\frac{\ell}{\mu\,\sqrt{2}}\,, \qquad \sin\tilde\delta = \frac{k}{\mu\,\sqrt{2}}\,, \tag{34}$$

where $\sqrt{2}\,\mu \equiv \sqrt{k^2+\ell^2}$.

In what follows we shall use real left-invariant 1-forms defined by $\sigma = \sigma_1 + \mathrm{i}\,\sigma_2$, $\Sigma = \Sigma_1 + \mathrm{i}\,\Sigma_2$ and $\nu = \nu_1 + \mathrm{i}\,\nu_2$. It was shown in [10] that if one defines 8-metrics of cohomogeneity one as follows:

$$ds_8^2 = dt^2 + a^2\,\sigma_i^2 + b^2\,\Sigma_i^2 + c^2\,\nu_i^2 + f^2\,\lambda^2\,, \tag{35}$$

where a, b, c and f are functions of the radial coordinate t, then the first-order equations

$$\dot a \;=\; \frac{b^2+c^2-a^2}{b\,c} - \frac{\sqrt{2}\,f\,\cos\tilde\delta}{a}\,,$$

$$
\begin{aligned}
\dot{b} &= \frac{a^2+c^2-b^2}{c\,a} + \frac{\sqrt{2}\,f\,\sin\tilde{\delta}}{b}\,,\\
\dot{c} &= \frac{a^2+b^2-c^2}{a\,b} + \frac{\sqrt{2}\,f\,(\cos\tilde{\delta}-\sin\tilde{\delta})}{c}\,,\\
\dot{f} &= -\frac{\sqrt{2}\,f^2\,(\cos\tilde{\delta}-\sin\tilde{\delta})}{c^2} + \frac{\sqrt{2}\,f^2\,\cos\tilde{\delta}}{a^2} - \frac{\sqrt{2}\,f^2\,\sin\tilde{\delta}}{b^2}\,,
\end{aligned} \tag{36}
$$

are the integrability conditions for Spin(7) holonomy.

We can study Einstein 7-metrics on the principal orbits by taking

$$
a=\bar{a}\,t\,,\quad b=\bar{b}\,t\,,\quad c=\bar{c}\,t\,,\quad f=\bar{f}\,t\,, \tag{37}
$$

and solving the equations for the constants $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{f}$ that result from substituting (37) into the first-order equations (36). In other words, since we then have

$$
ds_8^2 = dt^2 + t^2\,ds_7^2\,, \tag{38}
$$

with

$$
ds_7^2 = \bar{a}^2\,\sigma_i^2 + \bar{b}^2\,\Sigma_i^2 + \bar{c}^2\,\nu_i^2 + \bar{f}^2\,\lambda^2\,, \tag{39}
$$

it must be that if the 8-metric is Ricci-flat, then ds_7^2 is Einstein, satisfying $R_{ab} = 6\,g_{ab}$. Furthermore, since the first-order equations are the conditions for ds_8^2 to have Spin(7) holonomy, it follows that ds_7^2 will have weak G_2 holonomy; in other words it will not only be Einstein, but it will admit a Killing spinor. Since the results of [8] showed that the Einstein metrics discussed there admitted one or more Killing spinors, and the results of [9] showed that all of the Einstein metrics on the $N(k,\ell)$ spaces admit one or more Killing spinors, we will not be losing any generality in our construction of Einstein metrics on $N(k,\ell)$ by imposing the additional requirement of weak G_2 holonomy. We shall, however, have the advantage of having a simpler "first-order" system of equations to work with.

In order to simplify the notation, we shall drop the bar symbols from the constants in the 7-metric (39). Thus after making the substitution, we find that the metric

$$
ds_7^2 = a^2\,\sigma_i^2 + b^2\,\Sigma_i^2 + c^2\,\nu_i^2 + f^2\,\lambda^2\,, \tag{40}
$$

is Einstein, satisfying $R_{ab} = 6g_{ab}$, and of weak G_2 holonomy, if the constants a, b, c and f satisfy the conditions

$$
\begin{aligned}
\frac{\ell\,f}{\mu\,a^2} &= \frac{b^2+c^2-a^2}{a\,b\,c} - 1\,,\\
\frac{k\,f}{\mu\,b^2} &= \frac{c^2+a^2-b^2}{a\,b\,c} - 1\,,
\end{aligned}
$$

$$
\begin{aligned}
\frac{m\,f}{\mu\,c^2} &= \frac{a^2+b^2-c^2}{a\,b\,c} - 1\,,\\
\Big(\frac{\ell}{a^2}+\frac{k}{b^2}+\frac{m}{c^2}\Big)\,\frac{f}{\mu} &= 1\,,
\end{aligned}
\tag{41}
$$

where in order to emphasise the symmetry, we have defined $m \equiv -k-\ell$. In fact, the system is invariant under the simultaneous action of the permutation group S_3 on (ℓ,k,m) and (a,b,c).

The permutation group S_3 can be generated by two Z_2 elements, namely

$$
\begin{aligned}
A: &\qquad k \longrightarrow \ell\,,\quad \ell \longrightarrow k\,,\quad m \longrightarrow m\,,\\
B: &\qquad k \longrightarrow k\,,\quad \ell \longrightarrow m\,,\quad m \longrightarrow \ell\,.
\end{aligned}
\tag{42}
$$

If we define $x \equiv k/\ell$, then we shall have:

$$
\begin{aligned}
A: &\qquad x \longrightarrow \frac{1}{x}\,,\\
B: &\qquad x \longrightarrow -\frac{x}{1+x}\,.
\end{aligned}
\tag{43}
$$

It is easily seen that a "fundamental domain"

$$0 \le x \le 1 \tag{44}$$

can therefore be chosen, with all other values of $x = k/\ell$ obtainable from this by acting with the S_3 permutation group.

To solve the equations (41), we first note that two independent relations involving only a, b and c can be derived, one by adding all the equations, and the other by summing a^2 times the first, b^2 times the second and c^2 times the third. Thus we have

$$a^2+b^2+c^2 = 4a\,b\,c\,,\qquad a^4+b^4+c^4 = 6a^2\,b^2\,c^2\,. \tag{45}$$

It is straightforward to see that the general solution to these equations can be written in terms of an angle ϕ, such that

$$a^2 = \frac{(2+\cos\phi)^2}{2(3+2\cos\phi+\sin\phi)}\,,\qquad b^2 = \frac{(2+\cos\phi)^2}{2(3+2\cos\phi-\sin\phi)}\,, \tag{46}$$

where

$$0 \le \phi < 2\pi\,. \tag{47}$$

Substituting back into the remaining equations we therefore have

$$
\begin{aligned}
&\frac{k}{\ell} = \frac{4+6\cos\phi+12\sin\phi+5\sin 2\phi}{4+6\cos\phi-12\sin\phi-5\sin 2\phi}\,,\qquad c^2 = \frac{(2+\cos\phi)^2}{8+12\cos\phi+5\cos^2\phi}\,,\\
&f^2 = \frac{(2+\cos\phi)^2\,(\cos 2\phi+25\cos^4\phi+60\cos^3\phi-72\cos\phi-39)}{4(8+12\cos\phi+5\cos^2\phi)^2}\,.
\end{aligned}
\tag{48}
$$

It should be recalled that we have normalised the Einstein metrics so that they all have $R_{ab}=6g_{ab}$.

Note that the set of solutions that we have obtained here maps into itself under the action of the S_3 permutation group. It is easily seen that the Z_2 transformation A in (43) is implemented by the replacement

$$\phi \longrightarrow \phi' = 2\pi - \phi\,, \tag{49}$$

and this interchanges a^2 and b^2 (as well as k and ℓ), while leaving c^2 fixed. The Z_2 transformation B in (43) is slightly trickier to implement. It is achieved by transforming from ϕ to ϕ' where

$$\cos\phi' = -\frac{2(1+\cos\phi-\sin\phi)}{3+2\cos\phi-\sin\phi}\,,\qquad \sin\phi' = \frac{1+2\cos\phi+\sin\phi}{3+2\cos\phi-\sin\phi}\,. \tag{50}$$

This interchanges a^2 and c^2 (as well as implementing the mapping on k/ℓ given in (43)), while leaving b^2 fixed.

As ϕ varies over its range specified in (47), the function k/ℓ traverses each point on the real line exactly twice. Of course the allowed values for ϕ are those for which the expression for k/ℓ in (48) are rational. In general, the two values ϕ_1 and ϕ_2 for ϕ that give the same k/ℓ lead to *inequivalent* sets of values for the constants a, b, c and f, and hence to inequivalent Einstein metrics. However, if it should happen that ϕ_1 and ϕ_2 are related by $\phi_2 = 2\pi - \phi_1$, then it is evident from (46) and (48) that the associated pair of solutions will be equivalent, with a and b interchanged. This occurs only when $k/\ell = -1$, and so in this case there is just one Einstein metric. (The two values of ϕ that give rise to $k/\ell = -1$ are $\phi_1 = \arccos(-\frac{2}{3})$ and $\phi_2 = 2\pi - \arccos(-\frac{2}{3})$.) Thus we have reproduced the result of [9], that each $N(k,\ell)$ space has two inequivalent Einstein metrics, except for $N(1,-1)$, which has only one.

One final point remains. We have defined the 1-form λ as in (32). Despite naive appearances, this means that it is not in fact normalised to a fixed length for arbitrary $\tilde\delta$. This is because the "metric" for calculating the length is not simply a 2×2 unit matrix in the $\mathbb{C}^2$ subspace spanned by ${L_1}^1$ and ${L_2}^2$. In fact, one should calculate lengths using the 3×3 unit matrix in the $\mathbb{C}^3$ spanned by ${L_1}^1$, ${L_2}^2$ and ${L_3}^3$, projected onto the plane defined by ${L_1}^1+{L_2}^2+{L_3}^3=0$ (the condition that ensures the ${L_A}^B$ are in $SU(3)$ and not $U(3)$).

The easiest way to calculate the length of λ is therefore to add the appropriate multiple of the $U(1)$ generator $U \equiv {L_1}^1+{L_2}^2+{L_3}^3$ which lies in $U(3)$ but not in $SU(3)$, such that the shifted 1-form $\tilde\lambda$ is orthogonal to U, which implies:

$$\tilde\lambda = \frac{\sqrt{2}}{3}\Big[(2\cos\tilde\delta-\sin\tilde\delta)\,{L_1}^1 + (2\sin\tilde\delta-\cos\tilde\delta)\,{L_2}^2 - (\cos\tilde\delta+\sin\tilde\delta)\,{L_3}^3\Big]\,. \tag{51}$$

Thus we see that the length of $\tilde\lambda$, and hence, by definition, the length of λ, is given by

$$|\lambda| = \tfrac{2}{\sqrt3}\,(1-\sin\tilde\delta\,\cos\tilde\delta)^{1/2}\,. \tag{52}$$

Finally, since the λ term appears in the metric via $ds_7^2 = f^2\,\lambda^2+\cdots$, and we now want to express this in terms of a universally-normalised quantity

$$\hat\lambda \equiv \frac{\lambda}{|\lambda|}\,, \tag{53}$$

so that $ds_7^2 = \hat f^2\,\hat\lambda^2+\cdots$, we see that we should define

$$\hat f^2 = \tfrac43(1-\sin\tilde\delta\,\cos\tilde\delta)\,f^2\,. \tag{54}$$

The quantity $\hat f$ will be invariant under the S_3 permutation group. Written in terms of k, ℓ and $m=-k-\ell$, we have

$$\hat f^2 = \tfrac43\,\frac{k^2+\ell^2+k\,\ell}{k^2+\ell^2}\,f^2 = \tfrac23\,\frac{k^2+\ell^2+m^2}{k^2+\ell^2}\,f^2\,. \tag{55}$$

The numerator factor $(k^2+\ell^2+m^2)$ is clearly invariant under the permutation group, but the denominator $(k^2+\ell^2)$ is not. It is this non-symmetric denominator that corrects for the non-permutation-invariance of f, making $\hat f$ permutation invariant. In fact we can see that if f is replaced by $\hat f$ in (41), then the factors of $\mu=\sqrt{k^2+\ell^2}/\sqrt2$ in the denominators are precisely removed, so that the equations become manifestly permutation symmetric, with $\hat f$ being invariant.

It should be emphasised that using the Cartan-Maurer equation $dL_A{}^B = {\rm i}\,A_A{}^C\wedge L_C{}^B$ we have $dU=0$, and so therefore $d\lambda$ and $d\tilde\lambda$ are *identical*. Thus there is nothing wrong with our using λ in our metric constructions, it is just that its length is not given by the expression one would naively have expected, but rather, by (52).

3.1.2 Einstein metrics on $N(k,\ell)$ from second-order equations

Having constructed the Einstein metrics on $N(k,\ell)$ from the first-order equations implying weak G_2 holonomy, it is instructive now to re-examine the second-order Einstein equations themselves. In fact, as we shall show, these imply the previous first-order equations, thus supplying another proof of the result in [9] that all the Einstein metrics on $N(k,\ell)$ have weak G_2 holonomy, and thus admit at least one Killing spinor.

By following the same strategy as in the previous subsection, but now calculating the conditions for Ricci-flatness of the cone over $N(k,\ell)$, we find that the metric (40) $N(k,\ell)$

will be Einstein, with Ricci tensor normalised to $R_{ab} = 6g_{ab}$, if the constants a, b, c and f satisfy

$$\begin{aligned} \frac{2f^2\,\ell^2}{\mu^2\,a^4} &= -6 + \frac{6}{a^2} + \frac{a^4 - b^4 - c^4}{a^2\,b^2\,c^2}\,, \\ \frac{2f^2\,k^2}{\mu^2\,b^4} &= -6 + \frac{6}{b^2} + \frac{b^4 - c^4 - a^4}{a^2\,b^2\,c^2}\,, \\ \frac{2f^2\,m^2}{\mu^2\,c^4} &= -6 + \frac{6}{c^2} + \frac{c^4 - a^4 - b^4}{a^2\,b^2\,c^2}\,, \\ \frac{2f^2}{\mu^2}\Big(\frac{\ell^2}{a^4} + \frac{k^2}{b^4} + \frac{m^2}{c^4}\Big) &= 6\,. \end{aligned} \tag{56}$$

The approach to solving these equations that we shall present here is an elaboration of the method that was presumably used in [8]. Since a fully explicit derivation was not included there, we shall give rather detailed results. We begin by introducing new variables as in [8]:

$$A = \frac{a^2}{c^2}\,, \quad B = \frac{b^2}{c^2}\,, \quad u = \frac{\sqrt{2}\,k\,f\,a}{\mu\,b\,c}\,, \quad v = \frac{\sqrt{2}\,\ell\,f\,b}{\mu\,a\,c}\,, \quad \lambda = \frac{6a^2\,b^2}{c^2}\,. \tag{57}$$

In terms of these, the equations (56) become

$$\begin{aligned} &6A + B^2 - A^2 - 1 - u^2 = \lambda\,, \quad 6B + A^2 - B^2 - 1 - v^2 = \lambda\,, \\ &6A\,B - A^2 - B^2 - (A\,v + B\,u)^2 + 1 = \lambda\,, \quad u^2 + v^2 + (A\,v + B\,u)^2 = \lambda\,. \end{aligned} \tag{58}$$

From these, we can obtain the following three equations in which λ is eliminated:

$$\begin{aligned} (2 + B^2)\,u^2 + 2A\,B\,u\,v + (1 + A^2)\,v^2 &= 6A + B^2 - A^2 - 1\,, \\ (1 + B^2)\,u^2 + 2A\,B\,u\,v + (2 + A^2)\,v^2 &= 6B + A^2 - B^2 - 1\,, \\ (1 + 2B^2)\,u^2 + 4A\,B\,u\,v + (1 + 2A^2)\,v^2 &= 6A\,B - A^2 - B^2 + 1\,. \end{aligned} \tag{59}$$

These can be viewed as three linear equations for the three quantities $x \equiv u^2$, $y \equiv v^2$ and $z \equiv u\,v$. After obtaining the resulting expressions for x, y and z in terms of A and B, we then recall that $x\,y = z^2$, which leads to a polynomial constraint of the form $P(A, B) = 0$. In fact, we find that

$$P(A, B) = \Big[(A + B - 3)^2 + 4(A - B)^2 - 4\Big]\,Q(A, B)\,, \tag{60}$$

where $Q(A, B)$ is a sixth-order polynomial that can be shown not vanish for any real positive A and B (see Appendix C for the expression for $Q(A, B)$, and the proof of its positivity for nonvanishing A and B). Thus from $P(A, B) = 0$ we conclude that

$$(A + B - 3)^2 + 4(A - B)^2 = 4\,, \tag{61}$$

which can be solved by writing $A+B-3=2\cos\phi$, $B-A=\sin\phi$, and hence

$$A=\tfrac{3}{2}+\tfrac{1}{2}\cos\phi-\sin\phi\,,\qquad B=\tfrac{3}{2}+\tfrac{1}{2}\cos\phi+\sin\phi\,. \tag{62}$$

The solutions for u and v then follow, giving

$$u=\frac{1}{\sqrt{2}}\,(\cos\phi+2\sin\phi)\,,\qquad v=\frac{1}{\sqrt{2}}(\cos\phi-2\sin\phi)\,. \tag{63}$$

Finally, we find that

$$\lambda=\tfrac{3}{2}(2+\cos\phi)^2\,, \tag{64}$$

and hence from (57) we can obtain expressions for a, b, c and f. These are in fact precisely the ones given in (46) and (48), which we previously obtained by solving the conditions for weak G_2 holonomy.

In summary, we have seen that the conditions (56) following from imposing the Einstein equations have precisely the same solution set as those coming from the simpler equations (41) that arose by requiring weak G_2 holonomy.

3.1.3 Global considerations

In order to investigate the global structure of the Spin(7) metrics that we shall construct later, it is important to understand it first in the Aloff-Wallach spaces themselves. In particular, since in most of our Spin(7) examples that we shall discuss below there will be a $\mathbb{CP}^2$ degenerate orbit at short distance, it is important to understand the structure of the Aloff-Wallach spaces *qua* bundles over $\mathbb{CP}^2$. Not surprisingly, triality plays an important role in this question, and in fact a generic space $N(k,\ell)$ can be viewed as any of three inequivalent such bundles.

An example that is rather familiar is the case of the $N(1,1)$ space. It is well known that $N(1,1)$ can be viewed as an $\mathbb{RP}^3$ bundle over $\mathbb{CP}^2$ (a physicist's discussion of this can be found in [9]). On the other hand, the principal orbits in the Calabi metric on $T^*\mathbb{CP}^2$ are also the same Aloff-Wallach space, and so clearly here it is being viewed as an $SU(2)$ bundle over $\mathbb{CP}^2$, since the degeneration to the $\mathbb{CP}^2$ orbit in the Calabi metric is a regular one, with the metric approaching $\mathbb{R}^4\times\mathbb{CP}^2$ locally.

In fact in general it can be shown that if we view the σ_i and Σ_i 1-forms in (40) as spanning the $\mathbb{CP}^2$ base, and ν_i and λ as spanning the 3-dimensional fibres, then the space $N(k,\ell)$ can be described as an S^3/Z_p lens-space bundle over $\mathbb{CP}^2$, where

$$p=|k+\ell|\,. \tag{65}$$

(Note that S^3/Z_0 is a degenerate example, for which the fibres will be $S^1 \times S^2$.)

Applied to the case of $N(1,1)$ and its cousins $N(1,-2)$ and $N(-2,1)$, we see that with respect to this convention choice of having the ν_i, together with λ, spanning the fibres, we find that $N(1,1)$ will be an $S^3/Z_2 = \mathbb{RP}^3$ bundle over $\mathbb{CP}^2$, while $N(1,-2)$ and $N(-2,1)$ will be S^3 bundles over $\mathbb{CP}^2$. This is consistent with the observations made above.

There are various ways of proving the above results about the topology of the bundles.[3] Here, we shall present a rather intuitive approach, base on a consideration of an explicit parameterisation of $SU(3)$, which is presented in Appendix A.

We begin by recalling that $SU(3)/\mathbb{Z}_3$ acts effectively on $\mathbb{CP}^2$, with stabiliser $U(2)$. Explicitly, if (Z^1, Z^2, Z^3) are homogeneous coordinates and $\zeta^1 = Z^1/Z^3, \zeta^2 = Z^2/Z^3$ are inhomogeneous coordinates on $\mathbb{CP}^2$, then we may express almost every element of $SU(2)$ as in appendix A, so that $(\phi, \theta, \psi, \xi)$ parameterise $\mathbb{CP}^2$, considered as the set of right cosets, and $(\tilde\phi, \tilde\theta, \tilde\psi, \tau)$ parameterise the $U(2)$ stabiliser of the origin, $(0,0,1)$. Note that inhomogeneous coordinates (ζ^1, ζ^2) are functions just of $(\phi, \theta, \psi, \xi)$, and conversely. The $U(2)$ stabiliser of the origin is $\tilde U(\tilde\phi, \tilde\theta, \tilde\psi)\,\exp(\mathrm{i}\sqrt{3}\tau\,\lambda_8)$ where the range of the angles is $\tilde\phi \in (0, 2\pi]$, $\tilde{t}heta \in [0,\pi]$, $\tilde\psi \in (0, 4\pi]$ and $\tau \in (0, 2\pi]$. The two coordinates $(\tau, \tilde\psi)$ label points in a maximal torus of $U(2)$. A fundamental domain for the torus is given by a rectangle in τ-$\tilde\psi$ space of width 2π and height 4π.

The circle $S^1_{k,\ell}$, parameterised by an angle α, may be expressed as

$$\exp \frac{\mathrm{i}\alpha}{2}\{(k-\ell)\,\lambda_3 + (k+\ell)\sqrt{3}\lambda_8\}\,, \tag{66}$$

and acting on the right it induces the action

$$\tau \to \tau + (k-\ell)\,\alpha\,, \qquad \tilde\psi \to \tilde\psi + (k-\ell)\,\alpha\,. \tag{67}$$

If $k+\ell \neq 0$ (which will be treated separately), we may define a coordinate Ψ which is invariant under the circle action, and which may be used to label its orbits, by

$$\tilde\psi\,\lambda_3 + \tau\,\sqrt{3}\,\lambda_8 = \frac{k-\ell}{k+\ell}\,\tau\,\lambda_3 + \frac{k+\ell}{k+\ell}\,\tau\,\sqrt{3}\,\lambda_8 + \Psi\,\lambda_3\,. \tag{68}$$

One has

$$\Psi = \tilde\psi - \frac{k-l}{k+l}\,\tau\,. \tag{69}$$

The problem is now to find the correct period for the angle Ψ. This leads to a picture of $N(k,\ell)$ as a lens space bundle over $\mathbb{CP}^2$. The period is determined by the requirement that

[3] We are very grateful to James Sparks and Nigel Hitchin for extensive help and discussions, and for explaining how the result described above arises.

as Ψ ranges over its allowed values, it labels uniquely every orbit of $S^1_{k,\ell}$ in $SU(3)$. To see how this is done it is helpful to consider some examples.

Let us consider $N(2,-1)$, for which we shall have $\Psi = \tilde{\psi} - 3\tau$. By examining the torus of side $2\pi \times 4\pi$ in $(\tau, \tilde{\psi})$ space and following the orbit passing through $(0,0)$ and its neighbours, it is easy to see that every orbit passes once and only once through a strip of width $\frac{4\pi}{3}$ in τ bounded by the straight lines $\Psi = 0$ and $\Psi = -4\pi$. (The verification of these facts is greatly assisted by drawing a diagram.)

As another example, consider $N(1,0)$, for which we shall have $\Psi = \tilde{\psi} - \tau$. Each orbit on the torus passes once and only once through the square subdomain $0 \le \tau \le 2\pi$, $0 \le \tilde{\psi}_{(2)}p$. The square lies inside the region bounded by straight line $\Psi = 2\pi$ and $\Psi = -2\pi$. Thus again the range of Ψ is 4π.

As a third example, consider $N(3,2)$, which will give $\Psi = \tilde{\psi} - \frac{\tau}{5}$. Following the orbit through the origin around the torus we see that since it winds around the τ direction 10 times for every winding around the $\tilde{\psi}$ direction. Thus the fundamental domain decomposes into ten strips of height $\frac{2\pi}{5}$ in $\tilde{\psi}$, and every orbit visits each such strip once and only once. The range of Ψ is therefore $\frac{2\pi}{5}$. By applying similar arguments, one can fairly easily see that in general, for $N(k,\ell)$, the period of Ψ will be $4\pi/|k+\ell|$.

It is also of interest to identify what the bundles are. For example, we can think of $N(2,-1)$ or $N(1,-2)$ as the bundle of unit cotangent vectors of $\mathbb{CP}^2$, i.e. $ST^\star\mathbb{CP}^2$. To see that $SU(3)$ acts transitively, we need only remark that the stabiliser $U(2) \subset SO(4)$ of a point in $\mathbb{CP}^2$ acts on the unit sphere in $\mathbb{R}^4$. We thus need to identify the stabiliser. In fact $SU(2) \subset U(2) \subset SO(4)$ acts simply-transitively, and the stabiliser is the circle action generated by an overall phase. In terms of inhomogeneous coordinates, the action of $S^1_{k,\ell}$ is $(\zeta^1, \zeta^2) \to (\exp \mathrm{i}\theta(2k+\ell)\, \zeta^1, \exp \mathrm{i}\theta\,(2\ell+k)\, \zeta^2)$. It follows that the cotangent vector at the origin of the form $(d\zeta^1, 0)$ will be left invariant if and only if $2k+\ell = 0$. In other words $S^1_{1,-2}$ is the stabiliser of cotangent vectors at the origin with vanishing second component. Similarly $S^1_{2,-1}$ is the stabiliser of unit cotangent vectors at the origin with vanishing first component. This is consistent with our result above that in the case of $N(2,-1)$, the period of Ψ is 4π.

We have seen above that $N(2,-1)$ and $N(1,0)$ correspond to the same bundle, since the period of Ψ is the same. This seems to be related to the following: We can think of the cotangent bundle of $\mathbb{CP}^2$ as the bundle of holomorphic 1-forms $\Lambda^{1,0}$. Now $\mathbb{CP}^2$ has no Spin(4) structure, but it does have a Spin$^c(4)$ structure. One may identify the Spin$^c(4)$ bundle with holomorphic forms $\Lambda^{\star,0}$. Under this identification, chirality corresponds to

Hodge duality. Thus the odd forms correspond to negative chirality spinors. It seems therefore that we may think of both $N(2,-1)$ and $N(1,0)$ in terms of the the bundle of unit negative-chirality spinors. The positive-chirality spinors correspond to even forms, $\Lambda^{0,0} \oplus \Lambda^{2,0}$. However, the even forms are left invariant by the $SU(2)$ subgroup of the $U(2)$ stabiliser, and so even if normalised to have unit length they cannot be a homogeneous space with respect to $SU(2)$. On the other hand, we can consider the bundle of suitably charged negative-chirality spinors. This amounts to giving the spinors a charge with respect to the connection whose curvature is the Kähler form.

To summarise, we see that in general $U(2)/S^1_{k,\ell}$ is a lens space of the form $L(1,N) \equiv S^3/\mathbb{Z}_N$, where $N = |k+\ell|$. For each geometrically distinct $N(k,\ell)$ space we will obtain in general three different lens spaces, corresponding to the action of the Weyl group S_3 of $SU(3)$. In particular cases we obtain fewer than three bundles. Thus for example $N(1,1)$ gives an $SO(3) \equiv \mathbb{RP}^3$ bundle. On the other hand its Weyl cousins $N(-1,2)$ and $N(2,-1)$ are both $SU(2) \equiv S^3$ bundles.

3.2 An explicit Spin(7) solution for all $N(k,\ell)$

Although we have not been able to obtain the general solution of the first-order equations (36) for Spin(7) metrics with $N(k,\ell)$ principal orbits, we have succeeded in finding an isolated exact solution to these equations for generic k and ℓ. To construct the solution, it is convenient to introduce a constant $\gamma \equiv \tan\tilde{\delta} = -k/\ell$. We find that there exists a solution in which the following algebraic relation among the metric functions a, b and c holds:

$$X \equiv (1-2\gamma)\, a^2 + (\gamma-2)\, b^2 + (1+\gamma)\, c^2 + y = 0\,, \tag{70}$$

where y is a constant that sets the scale of the solution. We shall choose y to be

$$y = 8(\gamma-2)\,(\gamma+1)\,(2\gamma-1) = \frac{8}{\ell^3}(k-\ell)(\ell-m)(m-k)\,, \tag{71}$$

where as usual $m \equiv -k-\ell$. By differentiating (70) and using the first-order equations, one obtains another algebraic equation, which we may call $Y \equiv \dot{X} = 0$. This again involves only a, b and c, but not f. Differentiating again, and using the first-order equations, gives $W \equiv \dot{Y} = 0$, which is an algebraic equation for a, b, c and f (linear in f). Thus from these equations we can solve for a, b and f in terms of c. Differentiating again, we get $Z \equiv \dot{W}$ which again must vanish for the solution. However, this must be satisfied identically, if the original supposition (70) is correct, since otherwise it would give us a solution for c as a pure constant. Calculation shows that indeed Z vanishes identically, so all is consistent, and the validity of imposing the relation (70) is established.

It turns out now to be advantageous to work with a function ρ, rather than c itself, in order to avoid square roots, where $c^2+9(\gamma-1)^2=\rho^2$. Thus the algebraic solutions for a, b, c and f in terms of ρ are

$$
\begin{aligned}
& a^2=(\rho-\gamma-1)\,(\rho-\gamma+5)\,, \quad b^2=(\rho+\gamma+1)\,(\rho-5\gamma+1)\,, \\
& c^2=(\rho-3\gamma+3)\,(\rho+3\gamma-3)\,, \\
& f^2=\frac{9(\gamma^2+1)\,(\rho-5\gamma+1)\,(\rho-\gamma+5)\,[\rho+3(\gamma-1)]}{2[\rho^2-(\gamma+1)^2]\,[\rho-3(\gamma-1)]}\,.
\end{aligned} \tag{72}
$$

We can now substitute these into any one of the first-order equations, in order to obtain the differential equation for ρ. (Since the algebraic relations above were obtained by repeated use of the first-order equations, there is only one remaining equation's worth of information to be extracted from the entire first-order system, so we can choose whichever of the four equations is most convenient. The $\dot{f}$ equation is a convenient choice.) Using the coordinate gauge choice $dt=f^{-1}\,dr$, we find that the differential equation for ρ is simply

$$
\rho'=\frac{\sqrt{2}}{3\sqrt{1+\gamma^2}}\,, \tag{73}
$$

whose solution may be taken to be

$$
\rho=\frac{\sqrt{2}}{3\sqrt{1+\gamma^2}}\,r\,. \tag{74}
$$

We see that ρ is essentially just the radial coordinate, and the metric can be written as

$$
ds^2=\tfrac{9}{2}(1+\gamma^2)\,\frac{d\rho^2}{f^2}+a^2\,(\sigma_1^2+\sigma_2^2)+b^2\,(\Sigma_1^2+\Sigma_2^2)+c^2\,(\nu_1^2+\nu_2^2)+f^2\,\lambda^2\,, \tag{75}
$$

where a, b, c and f are given by (72).

It is easy to see that the set of metrics we have obtained here maps into itself under triality. It is convenient to make use of this observation when analysing the global properties; it allows us to restrict attention to cases where the metric function c is the first of a, b and c to reach zero as ρ reduces from the asymptotic region at $\rho=\infty$. The vanishing of c will then signal the inner endpoint of the radial coordinate range. Before studying this endpoint in detail, we may first observe that at large distance the metrics are all asymptotically locally conical, since a, b and c grow linearly, while f tends to a constant.

If c is the first of a, b and c that vanishes as ρ reduces from infinity, say at $\rho=\rho_0$, then it must be that the factors in a^2 and b^2 in (72) are still all positive when ρ reaches ρ_0 from above. (We shall, without loss of generality, assume that the asymptotic region is where $\rho=+\infty$.) It is easy to see from (72) that for this to happen we must have

$$
\rho_0=-3\gamma+3\,,\qquad \gamma\le -\tfrac{1}{2}\,. \tag{76}
$$

This in turn means that we must have $k/\ell \geq \frac{1}{2}$, and since k and ℓ must therefore have the same sign, we may without loss of generality take them both non-negative. We therefore have

$$2k > \ell \geq 0\,. \tag{77}$$

Noting from (65) that the associated description of $N(k,\ell)$ will be as an S^3/Z_p bundle over $\mathbb{CP}^2$ with $p = k+\ell$, it follows that the case $p=1$ is achieved only if $\ell = 0$, $k = 1$. The conclusion from this is that the Spin(7) metrics (75) will have Z_p orbifold singularities on the $\mathbb{CP}^2$ bolt except in the case that the principal orbits are $N(1,0)$.[4]

If we consider a solution (72) and (75) for which k and ℓ satisfy the inequality (77), we shall have an ALC Spin(7) metric with $N(k,\ell)$ principal orbits, whose topology is an R^4/Z_p bundle over $\mathbb{CP}^2$ with $p = k+\ell$. Although this metric will be singular (if $p > 1$), it is a relatively "mild" singularity, in the sense that it is an orbifold in which the only infinities in the curvature will be delta-functions. Such spaces might in fact be relevant in string theory or M-theory, especially in view of the fact that M-theory reductions could only give chiral fermions if the reduction manifold is singular.

Another possibility is that the orbifold singularities could be resolved by considering more general metrics of higher cohomogeneity. A phenomenon of this sort is known to occur in four dimensions, with the multi-centre hyper-Kähler metrics. The N-centre multi-Eguchi-Hanson metric is complete and non-singular, and is asymptotic to the cone over S^3/Z_N. The $N=2$ example is nothing but the Eguchi-Hanson metric, which can be written in its familiar cohomogeneity one form. However, the higher-N metrics cannot have cohomogeneity one. Thus, for example, the metric one obtains by imposing the periodicity $4\pi/N$ on the Hopf fibre coordinate in the Eguchi-Hanson metric will have an orbifold singularity *qua* metric of the Eguchi-Hanson cohomogeneity-one type, if $N > 2$, but it nevertheless admits a perfectly non-singular resolution as an inhomogeneous multi-centre metric. It may be that a similar situation could arise with the resolution of metric on the cone over $N(k,\ell)$.

3.3 Small distance behaviour and numerical analysis

Since we have not been able to obtain the general solution to the first-order equations (36) analytically, we now turn to a numerical analysis. To begin we therefore need to construct Taylor expansion for the solutions that are regular at small distance, i.e. in the region

[4]The solution in this special case $N(1,0)$ has been obtained by [50], and we are very grateful to the authors for informing us of their result prior to publication. It has provided one of the motivations for our investigations in this section.

where one or more of the metric functions vanishes. For such an endpoint of the metric to be regular, it must be that the terms that approach zero must be associated with the collapse of spheres. In the present case, we find that the possibilities that may gave regular metrics are that f alone vanishes, corresponding to a collapse of circles, or else that f and a vanish, or f and b vanish, or f and c vanish, corresponding to a collapse of 3-spheres. (These last three are equivalent, modulo the S_3 permutation group.) One might also in principle have situations with collapsing 2-spheres (just a or just b or just c vanishing), or else with 5-spheres collapsing (by having (a, b, f) or (a, c, f) or (b, c, f) vanishing).

We shall first describe the set of triality-related cases giving collapsing 3-spheres (or lens spaces), where (c, f) or (a, f) or (b, f) vanish. Thus the bolt at short distance will be $\mathbb{CP}^2$. The idea is to obtain Taylor expansions up to tenth order or so, which can then be used in order to set initial data just outside the bolt, which can then be integrated numerically to infinity. We present just the first couple of orders in the Taylor expansions here.

3.3.1 A triality of short-distance solutions

Although we shall present the three possible classes of small-distance solution, corresponding to c or a or b vanishing together with f, it should be emphasised that it is really redundant to consider all three, since they are related by triality. Thus one can adopt two different viewpoints. One possibility is to stick with just one of the cases, say where c and f vanish, and then consider all possible $N(k, \ell)$ principal orbits, including, in particular, not only a given $N(k, \ell)$ but also its cousins $N(k, -k - \ell)$ and $N(-k - \ell, \ell)$. The other possibility is to consider all three cases, with (c, f), (a, f) or (b, f) vanishing, and then restrict attention to a "fundamental domain" among the $N(k, \ell)$ spaces, such as that defined in (44). Either viewpoint can be taken, but one should take care not "overcount" the possibilities by including all three cases and also including all the "cousins." In general, we shall find it convenient to adopt the first approach, and consider all the $N(k, \ell)$ cousins within the framework of just Case 1 below.

<u>Case 1</u>:

First we consider the short-distance Taylor expansion corresponding to the case where c and f vanish at $t = 0$. We find

$$a = 1 + \frac{5\cos\tilde{\delta} - 4\sin\tilde{\delta}}{6(\cos\tilde{\delta} - \sin\tilde{\delta})} t^2 + \cdots\,, \qquad b = 1 + \frac{4\cos\tilde{\delta} - 5\sin\tilde{\delta}}{6(\cos\tilde{\delta} - \sin\tilde{\delta})} t^2 + \cdots\,, \tag{78}$$

$$c = t + \tfrac{1}{\sqrt{2}}\Big(q(\cos\tilde{\delta} - \sin\tilde{\delta}) - \tfrac{1}{\sqrt{2}}\Big)\, t^3 + \cdots\,, \qquad f = -\frac{t}{\sqrt{2}(\cos\tilde{\delta} - \sin\tilde{\delta})} + q\, t^3 + \cdots \tag{79}$$

Case 2:

Now, we consider the triality-related case where it is instead a that vanishes along with f at $t=0$. We find

$$a = t - (\tfrac{1}{\sqrt{2}} q\, \cos\tilde{\delta} + \tfrac{1}{2})\, t^3 + \cdots\,, \qquad b = 1 + (\tfrac{2}{3} + \tfrac{1}{6}\tan\tilde{\delta})\, t^2 + \cdots\,,$$
$$c = 1 + (\tfrac{5}{6} - \tfrac{1}{6}\tan\tilde{\delta})\, t^2 + \cdots\,, \qquad f = \frac{t}{\sqrt{2}\cos\tilde{\delta}} + q\, t^3 + \cdots\,. \tag{80}$$

Case 3:

Finally, if b instead vanishes along with f, we get

$$a = 1 + (\tfrac{2}{3} + \tfrac{1}{6}\cot\tilde{\delta})\, t^2 + \cdots\,, \qquad b = t + (\tfrac{1}{\sqrt{2}} q\, \sin\tilde{\delta} - \tfrac{1}{2})\, t^3 \, + \cdots\,,$$
$$c = 1 + (\tfrac{5}{6} - \tfrac{1}{6}\cot\tilde{\delta})\, t^2 + \cdots\,, \qquad f = -\frac{t}{\sqrt{2}\sin\tilde{\delta}} + q\, t^3 + \cdots\,, \tag{81}$$

It is easy to see that the three cases listed above are related by triality.

Note that Case 1 is valid provided that $\cos\tilde{\delta} \ne \sin\tilde{\delta}$, and likewise that Case 2 is valid for $\cot\tilde{\delta} \ne 0$, and Case 3 for $\tan\tilde{\delta} \ne 0$. These exclusions are just a triality-related set. Following our policy of using just Case 1 for our discussion, we note that in this guise the excluded case is $N(1,-1)$. It is in fact easy to re-analyse the Taylor expansion in Case 1 when $\cos\tilde{\delta} = \sin\tilde{\delta}$; we find that we then get $f=0$ and the solution reduces to a Gromov-Hausdorff limit of S^1 times a 7-metric of G_2 holonomy, whose principal orbits are S^2 bundles over $\mathbb{CP}^2$. (G_2 metrics of this type will be discussed later, in section 5.1.) This is consistent with the general result discussed in section 3.1.3, where it was noted that the space $N(k,\ell)$ admits a description as an S^3/Z_p bundle over $\mathbb{CP}^2$, with $p=|k+\ell|$. Thus $N(1,-1)$ here corresponds to an S^3/Z_0 bundle over $\mathbb{CP}^2$. This bundle is a degenerate case, which is $S^1\times S^2$.

Before discussing the numerical integration of (36) using these small-distance Taylor expansions to set up the initial data outside the $\mathbb{CP}^2$ bolt, we first note that another situation of particular interest is when the principal orbits are $N(1,1)$, or its triality-related cousins $N(1,-2)$ or $N(-2,1)$. Studying these within the Case 1 framework, the example $N(1,1)$ (which is then viewed as an S^3/Z_2 (i.e. $\mathbb{RP}^3$) bundle over $\mathbb{CP}^2$, as can be seen from (65)) arises as the principal orbits in the Spin(7) manifold with Z_2 orbifold singularity that one gets by replacing S^4 by $\mathbb{CP}^2$ in the chiral spin bundle over S^4 whose Spin(7) metric was obtained in [1, 2]. Indeed, we find that the Taylor expansion in (79) gives this exact solution if we set $\tan\tilde{\delta} = -1$, and take the free parameter q to have the value $q=\frac{1}{6}$, which implies $a=b$ and $f=-c/2$.

The cousins $N(1,-2)$ and $N(-2,1)$ of $N(1,1)$ arise as the principal orbits in the hyper-Kähler Calabi metric on $T^*\mathbb{CP}^2$. As we shall discuss in more detail later, although this has the smaller holonomy $Sp(2) = \text{Spin}(5)$, it is in fact a particular solution of the Spin(7) first-order equations (36), in the case of $N(-2,1)$ and $N(1,-2)$. These correspond respectively to $\tan\tilde{\delta} = 2$ and $\tan\tilde{\delta} = \frac{1}{2}$. We find that the Taylor expansions (36) reduce to those for the exact Calabi solution if the free parameter q is chosen as follows:

$$\begin{aligned} \tan\tilde{\delta} = 2\,, \quad q = -\tfrac{\sqrt{10}}{3}: \qquad & a^2 + c^2 = b^2\,, \qquad a\,c = \sqrt{\tfrac{2}{5}}\,b\,f\,, \\ \tan\tilde{\delta} = \tfrac{1}{2}\,, \quad q = \tfrac{\sqrt{10}}{3}: \qquad & b^2 + c^2 = a^2\,, \qquad b\,c = \sqrt{\tfrac{2}{5}}\,a\,f\,. \end{aligned} \tag{82}$$

3.3.2 Results of numerical analysis

We are now in a position to make use of the series expansions of section 3.3.1 to provide initial data just outside the $\mathbb{CP}^2$ bolt, in order to perform a numerical integration of the first-order equations (36). Again, because of the triality, we need only discuss the series solution in Case 1, provided that we consider $N(k,\ell)$ for all k and ℓ. The discussion can be further narrowed since the Case 1 is invariant under the Z_2 symmetry $k \leftrightarrow \ell$, $a \leftrightarrow b$, $f \leftrightarrow -f$ and $q \leftrightarrow -q$. It follows that we need only concentrate on the cases with $|k| \le |\ell|$, implying that we can consider the Case 1 solution with $|\tan\tilde{\delta}| \le 1$.

The following is a summary of our numerical findings:

(a) For each given $\tan\tilde{\delta} = -k/\ell$, there exist a $q_0 > 0$, such that for parameters $q \ge q_0$, the functions (a,b,c,f) are regular. In the limiting case where $q = q_0$, the metric is AC, with the Einstein metric on $N(k\ell)$ on the base of the cone being the one for which ϕ, defined in (48), lies in the interval $[0,\pi)$. For $q > q_0$, the metrics are ALC, with f becoming a finite constant at large distance. The precise value of q_0 for each $\tilde{\delta}$ is difficult to determine numerically. As we have seen previously, for $\tan\tilde{\delta} = -1$, we have $q_0 = \frac{1}{6}$; and for $\tan\tilde{\delta} = \frac{1}{2}$, we have $q_0 = \sqrt{10}/3$.

Thus we see that for a generic value of $\tilde{\delta}$, there exists an AC metric and a family of ALC metrics with a non-trivial parameter, which are all regular aside from having an $\mathbb{R}^4/Z_p$ orbifold singularity on the $\mathbb{CP}^2$ bolt, where $p = |k+\ell|$. In particular, this means that the AC metric and the ALC family of metrics are completely non-singular in the case of $N(k, 1-k)$ principal orbits, for all integers k.

(b) Without loss of generality, we can restrict the integers k and ℓ so that $0 < k \le l$, and then enumerate the Case 1 solutions for all three cousins $N(k,\ell)$, $N(k,-k-\ell)$ and $N(\ell,-k-\ell)$. For generic values of k and ℓ, the three cousins will give rise to three distinctly different sets of AC and ALC solutions. If we focus in particular on the AC solutions, then the choice $N(k,\ell)$ will be asymptotic to the cone over one of the two Einstein metrics on $N(k,\ell)$, whilst its two cousins $N(k,-k-\ell)$ and $N(\ell,-k-\ell)$ will give a pair of (inequivalent) AC solutions that are asymptotic to the cone over the *other* Einstein metric on this particular Aloff-Wallach space. This implies that each given asymptotic cone structure admits two different small-distance resolutions.

Case 4:

We have seen that the Case 1 solution is valid provided that $\tan\tilde{\delta} \neq 1$. When $\tan\tilde{\delta} = 1$ two possibilities arise, one of which is that $f = 0$, as we discussed earlier. Another possibility is that f, as well as a and b, becomes a constant at small distance. To the first few orders, we find that the solution is given by

$$\begin{aligned}
a &= 1 - \tfrac{1}{3}q\,t + (1 - \tfrac{5}{18}q^2)\,t^2 + (\tfrac{7}{45} - \tfrac{167}{810}q^2)\,q\,t^3 + \cdots\,, \\
b &= 1 + \tfrac{1}{3}q\,t + (1 - \tfrac{5}{18}q^2)\,t^2 - (\tfrac{7}{45} - \tfrac{167}{810}q^2)\,q\,t^3 + \cdots\,, \\
c &= 2t + \tfrac{4}{27}(q^2 - 9)\,t^3 + \cdots\,, \qquad f = q + \tfrac{2}{3}q^3\,t^2 + \cdots\,.
\end{aligned} \tag{83}$$

(Owing to triality, there are also two additional (equivalent) types of solution for either $\cos\tilde{\delta} = 0$ or $\sin\tilde{\delta} = 0$.)

A numerical analysis shows that there exist regular solutions for $|q| \le q_0 = 0.87\cdots$, such that the functions a, b, c and f are regular as one integrates outwards. When $|q| = q_0$, the solution is AC, whilst for $|q| < q_0$, we have a non-trivial 1-parameter family of ALC solutions. For $q = 0$, we recover the case with $f = 0$ mentioned above.

In this class of solutions, where only the metric function c vanishes at small distance, we see from (35) that we have collapsing 2-spheres with metric described by ν_i^2, whilst the terms in σ_i^2, Σ_i^2 and λ^2 describe homogeneous metrics on S^5 (viewed as an S^1 bundle over $\mathbb{CP}^2$). Thus we see that at short distance the metrics approach an $\mathbb{R}^3$ bundle over S^5. A straightforward calculation shows that the squashed metric

$$ds_5^2 = \sigma_i^2 + \Sigma_i^2 + x^2\,\lambda^2 \tag{84}$$

on S^5 becomes the standard $SO(6)$-invariant round metric if $x^2 = 1$. From (83) we see that our numerical result that $|q| \le q_0 = 0.87\cdots$ therefore means that all the regular examples arise when the $U(1)$ Hopf fibres on the S^5 bolt, viewed as an S^1 bundle over $\mathbb{CP}^2$, are squashed relative to their length in the round S^5 case.

4 Analytic results for the Spin(7) equations for $N(k,\ell)$ orbits

4.1 The general case $N(k,\ell)$

It is advantageous first to rescale the metric function f, in the fashion of (55), so that the rescaled function is a singlet under the S_3 permutation group. Accordingly, we shall define

$$\tilde f \equiv \frac{\sqrt{2}}{\sqrt{k^2+\ell^2}}\, f\,. \tag{85}$$

Next, we define new variables (A,B,F,G) in place of $(a,b,c,\tilde f)$ as follows:

$$A=\frac{a^2}{c^2}\,,\qquad B=\frac{b^2}{c^2}\,,\qquad F=\frac{\tilde f\, a\, b}{c^3}\,,\qquad G=\frac{a\,b}{c}\,. \tag{86}$$

These therefore satisfy the first-order equations

$$\begin{aligned}
\dot A &= \frac{1}{G}\,(4A-4A^2+2(k+\ell)\,A\,F+2\ell\,F)\,,\\
\dot B &= \frac{1}{G}\,(4B-4B^2+2(k+\ell)\,B\,F+2k\,F)\,,\\
\dot F &= \frac{F}{G}\,(5-3A-3B+4(k+\ell)\,F)\,,\\
\dot G &= 3-A-B+(k+\ell)\,F+\frac{\ell\,F}{A}+\frac{k\,F}{B}\,.
\end{aligned} \tag{87}$$

If we now introduce a new radial variable η, related to t by $dt=G\,d\eta$, these equations become

$$\begin{aligned}
A' &= 4A-4A^2+2(k+\ell)\,A\,F+2\ell\,F\,,\\
B' &= 4B-4B^2+2(k+\ell)\,B\,F+2k\,F\,,\\
F' &= (5-3A-3B)\,F+4(k+\ell)\,F^2\,,\\
G^{-1}\,G' &= 3-A-B+(k+\ell)\,F+\frac{\ell\,F}{A}+\frac{k\,F}{B}\,,
\end{aligned} \tag{88}$$

where a prime denotes a derivative with respect to η. We see that the first three equations now involve only the variables (A,B,F). From these, one can solve for F and A in terms of B, given by

$$\begin{aligned}
F &= \frac{B'-4B+4B^2}{2(k+(k+\ell)\,B)}\,,\\
A &= \tfrac13 (k+(k+\ell)\,B)^{-1}\,(B'-4B+4B^2)^{-1}\,\Big(-(k+(k+\ell)\,B)\,B''\\
&\quad (-11(2k+\ell)B+9(k+\ell)\,B^2+3(3k+(k+\ell)\,B'))\,B'\\
&\quad +4B\,(B-1)(5k-3(2k+\ell)\,B+5(k+\ell)\,B)\Big)\,.
\end{aligned} \tag{89}$$

The system then reduces down to the following non-linear differential equation for the function B:

$$\begin{aligned}
&18(k+\ell)\,B'^4 + 3(k+(k+\ell)B)^2\,(7B''^2 - 3(-4B+4B^2+B')\,B''') \\
&+12(2k(4k+5\ell) - (31k^2+42k\,\ell+11\ell^2)B + 3(k+\ell)^2\,B^2)\,B'^3 \\
&-64B\,(-k^2 - k\,\ell\,B - \ell\,(k+\ell)\,B + (k+\ell)\,B)\,B'' \\
&+4(k+(k+\ell)\,B)\,B')\Big((26k-2\ell)\,B + 12(k+\ell)\,B^2 - 3(6k+(k+ll)B')\Big)\,B'' \\
&8\Big(33k^2 - 9k\,(17k+6\ell)\,B + (260k^2+260k\,\ell+77\ell^2)B^2 \\
&-3(51k^2+92k\,\ell+41\ell^2)\,B^3 + 57(k+\ell)^2\,B^4\Big)\,B'^2 \\
&64B\,(B-1)\,\Big(12k^2 - 7k\,(7k+2\ell)\,B + 3(22k^2+22k\,\ell+7\ell^2)\,B^2 \\
&-7(7k^2+12k\,\ell+5\ell^2)\,B^3 + 12(k+\ell)^2\,B^4\Big)\,B' + 128B^2\,(B-1)^2\,\Big(5k^2 - 3k\,(7k+2\ell)\,B \\
&+(28k^2+28k\,\ell+9\ell^2)\,B^2 - 3(7k^2+12k\,\ell+5\ell^2)\,B^2 + 5(k+\ell)^2\,B^4\Big) = 0\,. \qquad (90)
\end{aligned}$$

Note that the equation is not explicitly dependent on the coordinate η, implying that we can reduce the system to a third-order equation by defining $a \equiv B$ and $b(a) \equiv B'$.

4.2 The case $N(1,-1)$

4.2.1 The first-order equations

Let us consider the Spin(7) equations specifically for the case of $N(1,-1)$ principal orbits. Setting $k = -\ell = 1$, we have

$$\begin{aligned}
\dot a &= \frac{b^2+c^2-a^2}{b\,c} - \frac{f}{a}\,, \qquad \dot b = \frac{c^2+a^2-b^2}{c\,a} + \frac{f}{b}\,, \\
\dot c &= \frac{a^2+b^2-c^2}{a\,b}\,, \qquad \dot f = \frac{f^2}{a^2} - \frac{f^2}{b^2}\,, \qquad (91)
\end{aligned}$$

and (88) then reduce to

$$\begin{aligned}
A' &= 4A - 4A^2 - 2F\,, \qquad B' = 4B - 4B^2 + 2F\,, \\
F' &= F\,(5 - 3A - 3B)\,, \qquad G^{-1}\,G' = 3 - A - B - \frac{F}{A} + \frac{F}{B}\,. \qquad (92)
\end{aligned}$$

From these equations, one can derive the following third-order equation for B:

$$\begin{aligned}
&128B^2\,(B^2-1)\,(9B^2-15B+5) + 88(7B^2-9B+3)\,B'^2 + 8(14B-9)\,B'\,B'' \\
&+64B\,(B-1)\,(21B^2-35B+12)\,B' - 64B\,(B-1)\,B'' + 7B''^2 - 24B'^2 \\
&-3[B' + 4B\,(B-1)]\,B''' = 0\,. \qquad (93)
\end{aligned}$$

If we now define a new variable Q by $Q \equiv \sqrt{9-8B}$, we may note that the following give solutions of (93):

$$\begin{aligned}
\textbf{(1)}: &\qquad Q' = \tfrac{1}{4}(Q^2-1)(Q-3)\,,\\
\textbf{(2)}: &\qquad Q' = \tfrac{1}{4}(Q^2-1)(Q+3)\,,\\
\textbf{(3)}: &\qquad Q' = \tfrac{1}{4}Q^{-1}\,(Q^2-1)(Q^2-9)\,.
\end{aligned} \tag{94}$$

Here a prime means $d/d\eta$. The solution (1) is

$$B = \frac{1+2e^{2\eta}+\sqrt{1+e^{2\eta}}}{2(1+e^{2\eta})}\,, \tag{95}$$

and it is the special case for $N(1,0)$ of the explicit solutions found in section 3.2, which was obtained in [50]. The solution (2) corresponds to interchanging the roles of A and B relative to (1), and it has

$$B = \frac{1+2e^{2\eta}-\sqrt{1+e^{2\eta}}}{2(1+e^{2\eta})}\,. \tag{96}$$

The solution (3) is rather trivial, and has

$$B = \frac{1}{1-e^{-4\eta}}\,, \tag{97}$$

which leads to $F=0$; it corresponds to a degeneration of the metric to a 7-dimensional one.

Another approach to solving the equations (92) is to define

$$A = X+Y\,,\quad B = X-Y\,. \tag{98}$$

The first three equations in (92) now give

$$X' = 4(X-X^2-Y^2)\,,\quad Y' = 4Y-8X\,Y-2F\,,\quad F' = (5-6X)\,F\,. \tag{99}$$

Calculating Y'', using the other first-order equations, and then using $d/d\eta = X'\,d/dX$, we get the second-order equation

$$\begin{aligned}
&(X-X^2-Y^2)\Big[4(X-X^2-Y^2)\,\frac{d^2Y}{dX^2} - 8Y\left(\frac{dY}{dX}\right)^2 + (6X-5)\,\frac{dY}{dX}\Big]\\
&+Y\,(5-8X+4X^2-8Y^2) = 0\,.
\end{aligned} \tag{100}$$

Note that a special solution of this equation is

$$Y = \frac{\sqrt{1-X}}{\sqrt{2}}\,, \tag{101}$$

which gives rise to the explicit metric in section 3.2 for the case $N(1,0)$, which was obtained in [50].

4.2.2 Heuristic discussion of the flows

We can give the following analysis of the fixed-points of the first-order equations (92). Solving for $A' = B' = F' = 0$, we see that the fixed points occur for (A, B, F) given by

$$(0,0,0)\,,\quad (0,1,0)\,,\quad (1,0,0)\,,\quad (1,1,0)\,,$$
$$(\tfrac{5+\sqrt{5}}{6}, \tfrac{5-\sqrt{5}}{6}, \tfrac{4\sqrt{5}}{9})\,,\quad (\tfrac{5-\sqrt{5}}{6}, \tfrac{5+\sqrt{5}}{6}, -\tfrac{4\sqrt{5}}{9})\,. \tag{102}$$

It is easily seen that $(0,0,0)$ is a degenerate point, $(1,0,0)$ and $(0,1,0)$ correspond to $\mathbb{CP}^2$ bolts at short distance, and $(1,1,0)$ is the large-distance asymptotic limit for ALC metrics. The two points $(\frac{5+\sqrt{5}}{6}, \frac{5-\sqrt{5}}{6}, \frac{4\sqrt{5}}{9})$ and $(\frac{5-\sqrt{5}}{6}, \frac{5+\sqrt{5}}{6}, -\frac{4\sqrt{5}}{9})$ correspond to large-distance asymptotic limits for AC metrics. The metrics on the principal orbits in these last two limits are precisely the Einstein metric on the $N(1,-1)$ Aloff-Wallach space, as discussed in section 5.1.

The explicit ALC solution found in [50] (described in section 3.2 for $N(1,0)$) corresponds to a flow from $(0,1,0)$ to $(1,1,0)$. As we adjust the non-trivial constant which parameterises inequivalent solutions of the first-order equations that start from a $\mathbb{CP}^2$ bolt at short distance, we get a family of flows that run from $(0,1,0)$ to the ALC endpoint at $(1,1,0)$. As the parameter is pushed to a limiting value, the distance at which the function f "turns over" and becomes asymptotically constant grows larger and larger. Eventually, at the limiting value of the parameter, the distance at which this happens gets pushed to infinity, and the endpoint jumps to the AC value $(\frac{5+\sqrt{5}}{6}, \frac{5-\sqrt{5}}{6}, \frac{4\sqrt{5}}{9})$. If the parameter is taken beyond the limiting value, the flow runs to some singular point and the metric is correspondingly singular.

The large-distance structure of the AC solution can be studied as follows. Let us suppose that we have the case $(\frac{5-\sqrt{5}}{6}, \frac{5+\sqrt{5}}{6}, -\frac{4\sqrt{5}}{9})$, in which B approaches $\frac{5+\sqrt{5}}{6}$ asymptotically. Setting

$$B = \tfrac{5+\sqrt{5}}{6} + y(\eta) \tag{103}$$

in (93), and then linearising in y, we obtain the third-order equation

$$9y''' + 48y'' - 16y' - 160y = 0\,. \tag{104}$$

Writing $y \sim e^{\lambda\, x}$, we find that the constant λ must satisfy the auxiliary equation

$$9\lambda^3 + 48\lambda^2 - 16\lambda - 160 = 0\,. \tag{105}$$

All three roots λ_i are real, with $\lambda_1 < 0$, $\lambda_2 < 0$ and $\lambda_3 > 0$. Since we want solutions that approach the AC limit (and hence $y \longrightarrow 0$ as $\eta \longrightarrow \infty$), we conclude that regular solutions must have the asymptotic form

$$y \sim x_1\, e^{\lambda_1\,\eta} + x_2\, e^{\lambda_2\,\eta}\,. \tag{106}$$

We can think of the general solution as being characterised by 3 parameters (excluding the completely trivial constant shift of η, but including the constant scaling). We see that the solutions with regular large-distance AC behaviour lie on a two-dimensional submanifold of ingoing trajectories, parameterised by the constants x_1 and x_2. On the other hand, we know that at the bolt, the solutions regular there also lie on a two-dimensional submanifold, of outgoing trajectories. Although we do not know analytically how to interpolate between the two regions, we can argue on general grounds that the intersection of the two-dimensional outgoing submanifold at short distance, and the two-dimensional AC ingoing submanifold at large distance, should occur along a curve. (This family would really be just a single non-trivial solution, since the single parameter along the curve would be a "trivial" one.) Thus we can expect a solution that is regular on the bolt and also regular at an AC infinity. This same conclusion is also indicated by the numerical solutions.

We can, of course, repeat the above discussion for the general case of $N(k,\ell)$ principal orbits. The principles are the same as for $N(1,-1)$ but the discussion is a little more involved since there are now two non-trivial fixed points that describe the flows to cones over the two inequivalent Einstein metrics on the Aloff-Wallach space. We find that at the linearised level, the analogue of (104) is now

$$y''' + 4y''\,(2+\cos\phi) - y'\,(2+\cos\phi)^2 - 4y\,(15\cos^3\phi + 20\cos^2\phi + 12\cos\phi + 8) = 0\,, \tag{107}$$

where ϕ is the angle parameterising the Einstein metrics on the Aloff-Wallach spaces, which was introduced in (46). The solutions will therefore be of the form $y \sim e^{\lambda\,\eta}$ with

$$\lambda^3 + 4\lambda^2\,(2+\cos\phi) - \lambda\,(2+\cos\phi)^2 - 4(15\cos^3\phi + 20\cos^2\phi + 12\cos\phi + 8) = 0\,. \tag{108}$$

It is easy to see that this cubic polynomial in λ has extrema at two values of λ, one negative and the other positive, for all values of ϕ. One can also see that the cubic is itself respectively positive and negative at the two extrema. This shows that all three roots λ_i of (108) are real, and that one, λ_1, is certainly negative, and another, λ_3, is certainly positive. Together with the fact that the cubic is negative at $\lambda = 0$, for all ϕ, we can deduce that the intervening root, λ_2, is negative, and so for all ϕ, two of the λ_i are negative and one is positive. Thus again we have a two-dimensional submanifold of ingoing trajectories, supporting the indications from the numerical analysis that there will be regular AC solutions.

4.3 Perturbative construction of AC metrics

We have obtained evidence by means of a numerical analysis that for each choice of $N(k,\ell)$ principal orbit, there are two possible AC Spin(7) metrics, which approach the cones over the two inequivalent Einstein metrics on $N(k,\ell)$. The only exception is $N(1,0)$, for which there is only one AC solution, since here there is only one possible Einstein metric on the base of the cone.

We have already alluded to the fact that for the special case of principal orbits that are $N(1,1)$, or its cousins $N(1,-2)$ and $N(-2,1)$, we actually know of two explicit AC solutions of the first-order equations (36). One such solution is the complete and non-singular hyper-Kähler Calabi metric on $T^*\mathbb{CP}^2$, which happens to have the smaller holonomy group $Sp(2)$, but nonetheless corresponds to a solution also of the Spin(7) first-order equations (36). If we make our usual choice where it is the function c in (5), rather than a or b, that vanishes at short distance, then the principal orbits will be $N(1,-2)$ or $N(-2,1)$ in this case. The other exact solution is the Spin(7) metric that one obtains by replacing S^4 by $\mathbb{CP}^2$ in the original construction in [1, 2] of the complete and non-singular Spin(7) AC metric on the chiral spin bundle of S^4. After the replacement, the metric will have a Z_2 orbifold singularity on the bolt, since now we have $N(1,1)$ as principal orbits, which can be described as an S^3/Z_p bundle over $\mathbb{CP}^2$ with $p=|k+\ell|=2$, i.e. an $SO(3)$ bundle over $\mathbb{CP}^2$. Nonetheless, the associated solution of the first-order equations is non-singular, and from a physical point of view in string theory, one might even find the orbifold singularity attractive.

Leaving aside for now the question of the acceptability or otherwise of an orbifold singularity, we can take the two exact solutions described in the previous paragraph as starting points for perturbative constructions of AC solutions of the first-order equations (36), for values of k/ℓ that are close to the values occurring in the exact solutions. Thus, for example, we can take the $N(1,1)$ solution with the Z_2 orbifold singularity, and then seek a solution with $k/\ell = 1-\epsilon$, order by order in ϵ. Of course we should ultimately have in mind that ϵ should be rational, but this does not present any difficulty.

The other starting point with an exact AC solution is the hyper-Kähler Calabi metric. In this particular instance we find it more convenient, rather than following our usual strategy of working with $N(1,-2)$ or $N(-2,1)$ principal orbits in the framework where c vanishes on the bolt, to work instead in the framework where a vanishes on the bolt, in which case we again have $N(1,1)$ principal orbits. Thus the perturbative expansions around both of the exact solutions can be parameterised by taking $\tan\tilde\delta = -1+\epsilon$.

Case (a): Expansion around $SO(3)$ bundle over $\mathbb{CP}^2$:

Here we take as our zeroth-order starting point the Spin(7) metric on the chiral spin bundle of S^4, given in [1, 2], but with S^4 replaced by $\mathbb{CP}^2$. The principal orbits are $N(1,1)$, with the metric function c^2 that multiplies ν_i^2 in (5) vanishing on the bolt. We shall work up to and including order ϵ^2 in the expansion around $k/\ell = 1$.

In order to simplify our results for the perturbative expansion it is helpful to introduce a new radial variable ρ, defined in terms of r by $\rho = r^{2/3}$. After some algebra, we find that the perturbative expansion up to order ϵ^2 is given by

$$\begin{aligned}
a &= \tfrac{3}{\sqrt{10}}\,\rho^{3/2}\Big(1 + \frac{3\rho^{11} - 11\rho^6 + 33\rho - 25}{264\rho\,(\rho^5-1)^2}\,(2\epsilon+\epsilon^2) + \mathcal{O}(\epsilon^3)\Big)\,,\\
b &= \tfrac{3}{\sqrt{10}}\,\rho^{3/2}\Big(1 - \frac{3\rho^{11} - 11\rho^6 + 33\rho - 25}{264\rho\,(\rho^5-1)^2}\,(2\epsilon+\epsilon^2) + \mathcal{O}(\epsilon^3)\Big)\,,\\
c &= \tfrac35\rho^{-1}\,(\rho^5-1)^{1/2}\,\Big(1 + c_2\,\epsilon^2 + \mathcal{O}(\epsilon^3)\Big)\,,\\
f &= -\tfrac{3}{10}\rho^{-1}\,(\rho^5-1)^{1/2}\,\Big(1 + f_2\,\epsilon^2 + \mathcal{O}(\epsilon^3)\Big)\,,\\
h &= \frac{\rho^{5/2}}{(\rho^5-1)^{1/2}}\,\Big(1 + h_2\,\epsilon^2 + \mathcal{O}(\epsilon^3)\Big)\,,
\end{aligned}\tag{109}$$

where

$$\begin{aligned}
c_2 &= \frac{(\rho-1)^4\,v_1}{2613600\rho^2\,(\rho^5-1)^5} + \rho^{-1}\,(\rho^5-1)\,u + (\rho^5-1)^{-1}\,\tilde u\,,\\
f_2 &= \frac{(\rho-1)^4\,v_2}{2613600\rho^2\,(\rho^5-1)^5} - \rho^{-1}\,(\rho^5-1)\,u + (\rho^5-1)^{-1}\,\tilde u\,,\\
h_2 &= \frac{(\rho-1)^4\,v_3}{2613600\rho^2\,(\rho^5-1)^5} - \tfrac13\rho^{-1}\,(\rho^5-1)\,u - (\rho^5-1)^{-1}\,\tilde u\,,
\end{aligned}\tag{110}$$

and the functions u and $\tilde u$ are given by

$$\begin{aligned}
u &= -\frac{7}{165\rho}\,{}_2F_1[1,\tfrac15,\tfrac65,\rho^{-5}] + \frac{7}{660\rho^4}\,{}_2F_1[1,\tfrac45,\tfrac95,\rho^{-5}]\,,\\
\tilde u &= k + \tfrac{7}{59400}\rho^3\,(15\rho^5 - 24\rho^2 - 40)\\
&\quad + \frac{7(\rho^5-1)^2}{495}\,\Big[\rho^{-2}\,{}_2F_1[1,\tfrac15,\tfrac65,\rho^{-5}] - \tfrac14\rho^{-5}\,{}_2F_1[1,\tfrac45,\tfrac95,\rho^{-5}]\Big]\\
&\quad - \tfrac{35}{4356}\Big[\log(\rho^5-1) + 10\rho^{-1}\,{}_2F_1[1,\tfrac15,\tfrac65,\rho^{-5}] - \tfrac52\rho^{-2}\,{}_2F_1[1,\tfrac25,\tfrac75,\rho^{-5}]\Big]\,.
\end{aligned}\tag{111}$$

Note that we have

$$\begin{aligned}
\frac{\partial u}{\partial\rho} &= \frac{7(\rho^3-1)}{165(\rho^5-1)}\,,\\
\frac{\partial\tilde u}{\partial\rho} &= \tfrac{7}{495}\rho^{-1} - \frac{175\rho^2\,(\rho-1)^2}{4356(\rho^5-1)} + \frac{(1+8\rho^5-9\rho^{10})\,u}{3\rho^2}\,.
\end{aligned}\tag{112}$$

The functions v_i are polynomials in ρ, given by

$$
\begin{aligned}
v_1 &= 31250 + 187500\rho + 873820\rho^2 + 2495280\rho^3 + 5456950\rho^4 + 9894075\rho^5 \\
&\quad +15688150\rho^6 + 21497477\rho^7 + 25980358\rho^8 + 27795095\rho^9 + 26221340\rho^{10} \\
&\quad +21387495\rho^{11} + 14948034\rho^{12} + 8557431\rho^{13} + 3870160\rho^{14} + 1498795\rho^{15} \\
&\quad +1344660\rho^{16} + 2431196\rho^{17} + 3781844\rho^{18} + 4420045\rho^{19} + 4281340\rho^{20} \\
&\quad +3511270\rho^{21} + 2437848\rho^{22} + 1389087\rho^{23} + 693000\rho^{24} + 277200\rho^{25} + 69300\rho^{26}\,, \\
v_2 &= -156250 - 502500\rho - 1192820\rho^2 - 2381280\rho^3 - 4221950\rho^4 - 6069165\rho^5 \\
&\quad -7561010\rho^6 - 7585363\rho^7 - 5030102\rho^8 + 1216895\rho^9 + 10072100\rho^{10} \\
&\quad +21255735\rho^{11} + 32192394\rho^{12} + 40306671\rho^{13} + 43023160\rho^{14} + 40712455\rho^{15} \\
&\quad +33048900\rho^{16} + 22612556\rho^{17} + 11983484\rho^{18} + 3741745\rho^{19} - 1745300\rho^{20} \\
&\quad -3900290\rho^{21} - 3838392\rho^{22} - 2674773\rho^{23} - 1524600\rho^{24} - 609840\rho^{25} - 152460\rho^{26}\,, \\
v_3 &= 125000 + 252500\rho + 38110\rho^2 - 862560\rho^3 - 2793900\rho^4 - 5655745\rho^5 \\
&\quad -10651680\rho^6 - 16695647\rho^7 - 22701588\rho^8 - 27583445\rho^9 - 29937510\rho^{10} \\
&\quad -28501325\rho^{11} - 24368354\rho^{12} - 18632061\rho^{13} - 12385910\rho^{14} - 7010765\rho^{15} \\
&\quad -3391240\rho^{16} - 1197516\rho^{17} - 99774\rho^{18} + 231805\rho^{19} + 256540\rho^{20} \\
&\quad +223750\rho^{21} + 160032\rho^{22} + 91983\rho^{23} + 46200\rho^{24} + 18480\rho^{25} + 4620\rho^{26}\,. \qquad (113)
\end{aligned}
$$

Here k in $\tilde{u}$ is an integration constant, which should be chosen to be

$$k = \tfrac{343}{59400} + \tfrac{35}{4356}\,\gamma + \tfrac{35}{4356}[2\psi(\tfrac{1}{5}) - \psi(\tfrac{2}{5})] \qquad (114)$$

in order that the solution be regular at small distance, where γ is the Euler-Mascheroni constant, and $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ is the digamma function.

At large distance, the functions become

$$
\begin{aligned}
a &= \tfrac{3}{\sqrt{10}}\, r\,(1 + \tfrac{1}{44}\epsilon + \tfrac{1}{88}\epsilon^2)\,, \qquad b = \tfrac{3}{\sqrt{10}}\, r\,(1 - \tfrac{1}{44}\epsilon - \tfrac{1}{88}\epsilon^2)\,, \\
c &= \tfrac{3}{5}\, r\,(1 + \tfrac{21}{3872}\,\epsilon^2)\,, \qquad f = -\tfrac{3}{10}\, r\,(1 + \tfrac{489}{3872}\,\epsilon^2)\,, \\
h &= 1 + \tfrac{31}{11616}\epsilon^2\,, \qquad (115)
\end{aligned}
$$

As a verification, one can check that the above cone metric matches precisely to the conifolds obtained in section 3, up to ϵ^2 order, expanding around $\phi = 0$. At small distance, it can be matched to the Case 1 in section 3.3, with the constant q specified as

$$q = \tfrac{1}{6} + \Big(\tfrac{2225}{26136} - \tfrac{7}{495}\,\pi\,\sqrt{1 + \tfrac{2}{\sqrt{5}}}\Big)\,\epsilon^2 + \mathcal{O}(\epsilon^3)\,. \qquad (116)$$

What we have seen emerging here is an orderly expansion of the metric functions around their unperturbed form, with corrections at order ϵ and ϵ^2 that are perfectly regular both at short distance and at large distance. This provides further evidence, of an analytical nature, for the existence of regular solutions of the first-order equations (36) for AC metrics (5) with Spin(7) holonomy, where the principal orbits are $N(k,\ell)$ with general values of k and ℓ. Of course one should distinguish between having regular solutions of the first-order equations, and having regular metrics, since, as we know, our starting point for the perturbation series in this case is a metric with a $\mathbb{Z}_2$ orbifold singularity. Thus our emphasis in this specific perturbation expansion is really with the regularity of the metric functions, rather than with the complete regularity of the 8-metrics. Nonetheless, as we mentioned previously, even those with orbifold singularities on the bolt may be of interest in string theory and M-theory. However, the main point emerging here is that we see strong supporting evidence for the proposition that there exist regular AC solutions of the first-order equations, for all k and ℓ. For those cases where $|k+\ell|=1$ (which are, of course, far away from the $\epsilon=0$ starting point here), we should therefore obtain complete and non-singular AC metrics.

Case (b): Expansion around Calabi metric:

In this case we take as our zeroth-order starting point the hyper-Kähler Calabi metric on $T^*\mathbb{CP}^2$, which is complete and non-singular. Since we shall choose to work in a framework where it is the metric function a that vanishes on the bolt, and since the S_3 symmetry of the system involves permuting (a,b,c) in step with (ℓ,k,m), where $m=-k-\ell$, it follows that instead of $N(k,\ell)$ being viewed as an S^3/Z_p bundle over $\mathbb{CP}^2$ with $p=|k+\ell|=|m|$ as it is when c vanishes on the bolt, we now have an S^3/Z_p bundle with $p=|\ell|$. Thus the non-singular hyper-Kähler Calabi metric is described in these conventions in terms of $N(1,1)$ or $N(-2,1)$ Aloff-Wallach spaces forming the principal orbits. We shall take $N(1,1)$, so that again our perturbation will be of the form $\tan\tilde\delta = 1-\epsilon$.

It should be emphasised that although our starting-point here is hyper-Kähler, we perturb around it using the usual Spin(7) first-order equations. Thus the reduced holonomy when $\epsilon=0$ is to be viewed as an "accidental" reduction that is a feature of this specific solution of the Spin(7) equations.

After some algebra, we find in this case that up to order ϵ^2, the perturbed solution is given by

$$a = \sqrt{\tfrac{1}{2}(r^2-1)}\,(1+a_1\,\epsilon+a_2\,\epsilon^2)\,,$$

$$
\begin{aligned}
b &= \sqrt{\tfrac{1}{2}(r^2+1)}\,(1+b_1\,\epsilon+b_2\,\epsilon^2)\,,\\
c &= r\,(1+c_1\,\epsilon+c_2\,\epsilon)\,,\\
f &= \tfrac{1}{2}r\sqrt{1-r^{-4}}(1+f_1\,\epsilon+f_2\,\epsilon^2)\,,\\
h &= \frac{r}{2f}\,,
\end{aligned}
\tag{117}
$$

where

$$
\begin{aligned}
a_1 &= \frac{1+3z-3z^2}{12z^2}+\frac{\log(z)}{6(z-1)}\,,\\
b_1 &= \frac{z-1}{4z}+\frac{\log z}{6z}\,,\\
c_1 &= \frac{(z-1)^2}{12z^2\,(2z-1)}+\frac{\log z}{3(2z-1)}\,,\\
f_1 &= \frac{4-9z}{12z\,(2z-1)}+\frac{(1-2z+2z^2)\,\log z}{6z\,(z-1)\,(2z-1)}\,,
\end{aligned}
\tag{118}
$$

and

$$
\begin{aligned}
a_2 &= \frac{15+268z-431z^2-272z^3+675z^4}{4320z^4}+\frac{7\psi(2,1-z)}{60(z-1)}\\
&\quad+\frac{(-20+26z+31z^2-97z^3+60z^4+z^3\,(21z-26)\,\log z)\,\log z}{360z^3\,(z-1)^2}\,,\\
b_2 &= \frac{16+38z-147z^2+989z^3-2903z^4+1755z^5}{4320z^4\,(z-1)}+\frac{7\psi(2,1-z)}{60z}\\
&\quad+\frac{(35-127z+233z^2-120z^3+(z-1)^2\,(21z-5)\,\log z)\,\log z}{360z^2\,(z-1)^2}\,,\\
c_2 &= \frac{-15+247z-1500z^2+4316z^3-7795z^4+10359z^5-8600z^6+3240z^7}{4320z^4\,(z-1)\,(2z-1)^2}\\
&\quad-\frac{(-20+131z-306z^2+274z^3+24z^4-202z^5+120z^6)\,\log z}{360z^3\,(z-1)^2\,(2z-1)^2}\\
&\quad+\frac{(42z-31)\,\log^2 z}{180(2z-1)}+\frac{7\psi(2,1-z)}{30(2z-1)}\\
f_2 &= -\frac{4+12z-324z^2+1216z^3+209z^4-4720z^5+3240z^6}{4320z^4\,(2z-1)^2}\\
&\quad+\frac{7(1-2z+2z^2)\,\psi(2,1-z)}{60z\,(z-1)\,(2z-1)}-\frac{(20-144z+351z^2-301z^3+60z^4)\,\log z}{180z^2\,(z-1)\,(2z-1)^2}\\
&\quad+\frac{(-5+61z-165z^2+250z^3-230z^4+84z^5)\,\log^2 z}{360z^2\,(z-1)^2\,(2z-1)^2}\,.
\end{aligned}
\tag{119}
$$

Here $z=\frac{1}{2}(1+r^2)$ and $\psi(2,x)\equiv-\int^z \tilde{z}^{-1}\,\log(1-\tilde{z})\,d\tilde{z}$ is the di-logarithm function.

At large distance, we have

$$
\begin{aligned}
&a=\frac{r}{\sqrt{2}}\Big(1-\tfrac{1}{4}\epsilon+\tfrac{5}{32}\epsilon^2\Big)\,,\qquad b=\frac{r}{\sqrt{2}}\Big(1+\tfrac{1}{4}\epsilon+\tfrac{13}{32}\epsilon^2\Big)\\
&c=(1+\tfrac{3}{16}\epsilon^2)\,r\,,\qquad f=\tfrac{1}{2}(1-\tfrac{3}{16}\epsilon^2)\,r\,,\qquad h=1+\frac{3}{16}\,\epsilon^2\,.
\end{aligned}
\tag{120}
$$

As a verification, one can check that the above cone metric matches precisely to the conifolds obtained in the previous section, up to ϵ^2 order, expanding around $\phi = \pi$.

At small distance, it can be matched with Case 2 in section 4.3, provided that $q = -2/3 + \epsilon - \frac{277}{270}\epsilon^2$.

Again, we are seeing that an orderly perturbative expansion can be developed, with corrections to the metric functions at order ϵ and ϵ^2 that are regular both at short distances and at large distances. This again provides analytical evidence supporting the findings from our numerical analysis, that regular AC solutions of the first-order equations (36) should exist for all k and ℓ.

5 More general 7-metrics of G_2 holonomy

Having studied more general cohomogeneity 8-metrics with Spin(7) holonomy, we now turn to the consideration of analogous generalisations for 7-metrics of G_2 holonomy.

5.1 New G_2 metrics on $\mathbb{R}^3$ bundle over $\mathbb{CP}^2$

We start from the left-invariant 1-forms $L_A{}^B$ of $SU(3)$, and define complex 1-forms $\sigma \equiv L_1{}^3$, $\Sigma \equiv L_2{}^3$ and $\nu \equiv L_1{}^2$, as in section 3.1.1. Defining real 1-forms via $\sigma = \sigma_1 + \mathrm{i}\,\sigma_2$, etc, we then make the ansatz

$$ds_7^2 = dt^2 + a^2\,\sigma_i^2 + b^2\,\Sigma_i^2 + c^2\,\nu_i^2\,. \tag{121}$$

This is very like the ansatz for eight-dimensional Spin(7) metrics in (35), except that the extra $U(1)$ direction $f^2\,\lambda^2$ in equation (3.2) there is dropped. We can therefore read off the results for curvature, T and V from section 2 of [10], and reproduced in (36), by dropping all the f terms. Thus we have

$$\begin{aligned} T &= 2\alpha'^2 + 2\beta'^2 + 2\gamma'^2 + 8\alpha'\,\beta' + 2\beta'\,\gamma' + 2\alpha'\,\gamma'\,, \\ V &= -\frac{12}{a^2} - \frac{12}{b^2} - \frac{12}{c^2} + \frac{2a^2}{b^2\,c^2} + \frac{2b^2}{a^2\,c^2} + \frac{2c^2}{a^2\,b^2}\,. \end{aligned} \tag{122}$$

Note that the principal orbits are the coset space $SU(3)/(U(1)\times U(1))$, which is the six-dimensional flag manifold.

We find that V can be derived from the superpotential

$$W = 4a\,b\,c\,(a^2 + b^2 + c^2)\,. \tag{123}$$

From this, we arrive at the first-order equations

$$\frac{\dot a}{a} = \frac{b^2 + c^2 - a^2}{a\,b\,c}\,,\qquad \frac{\dot b}{b} = \frac{c^2 + a^2 - b^2}{a\,b\,c}\,,\qquad \frac{\dot c}{c} = \frac{a^2 + b^2 - c^2}{a\,b\,c}\,. \tag{124}$$

It should be noted that these are identical to one of the sets of first-order equations that can be derived for the triaxial Bianchi IX system in $D=4$, with $ds_4^2 = dt^2 + a^2\,\sigma_1^2 + b^2\,\sigma_2^2 + c^2\,\sigma_3^2$, where here the σ_i are the left-invariant 1-forms of $SU(2)$. Specifically, they coincide with the $D=4$ equations that correspond to the Nahm equations for the "spinning top." This is the first-order system that admits Eguchi-Hanson as a non-singular solution if $a=b$. For unequal a, b and c, the system studied in [11], and the general solution was obtained. It was found that the associated Ricci-flat metrics were singular when the three functions were unequal.

We can use the same method here to solve the first-order equations (124). Thus we let $u = a\,b$, $v = b\,c$ and $w = c\,a$. After defining a new radial coordinate r by $dr = a\,b\,c\,dt = \sqrt{u\,v\,w}\,dt$, we then get

$$\frac{du}{dr} = \frac{2}{u}\,,\qquad \frac{dv}{dr} = \frac{2}{v}\,,\qquad \frac{dw}{dr} = \frac{2}{w}\,, \tag{125}$$

with the general solution

$$u^2 = 4(r-r_1)\,,\qquad v^2 = 4(r-r_2)\,,\qquad w^2 = 4(r-r_3)\,, \tag{126}$$

where r_1, r_2 and r_3 are constants of integration.[5] The metric is

$$ds_7^2 = \frac{dr^2}{u\,v\,w} + \frac{u\,w}{v}\,(\sigma_1^2+\sigma_2^2) + \frac{u\,v}{w}\,(\Sigma_1^2+\Sigma_2^2) + \frac{v\,w}{u}\,(\nu_1^2+\nu_2^2)\,. \tag{127}$$

It can be seen that this is singular unless two of the r_i are set equal. If two are set equal, so that $a=b$, we get, after a coordinate transformation, the previously-known G_2 metric on the $\mathbb{R}^3$ bundle over $\mathbb{CP}^2$.

The G_2 holonomy can be checked by looking for a covariantly-constant spinor. Equivalently, we can check to see if there is a covariantly-constant 3-form (the calibrating form). From the exterior derivatives of the complex 1-forms given in the hyper-Kähler paper, we can easily verify that

$$d(\sigma\wedge\bar\sigma) = -d(\Sigma\wedge\bar\Sigma) = d(\nu\wedge\bar\nu) = -2\mathrm{i}\,\Re(\bar\sigma\wedge\Sigma\wedge\nu)\,. \tag{128}$$

From this we see that the 3-form $G_{(3)}$, defined by

$$G_{(3)} \equiv a\,b\,c\,\Re(\bar\sigma\wedge\Sigma\wedge\nu) + \mathrm{i}\,(-a^2\,\sigma\wedge\bar\sigma + b^2\,\Sigma\wedge\bar\Sigma + c^2\,\nu\wedge\bar\nu) \tag{129}$$

is closed, $dG_{(3)}=0$, by virtue of the first-order equations (124). A more complete calculation should show that it is in fact covariantly constant.

Note that the vielbein components of $G_{(3)}$ will be constants.

[5] We understand that the first-order equations (124) for the Spin(7) metrics (121) have also been obtained independently by R. Cleyton (PhD thesis, Odense University), and by A. Dancer and M.Y. Wang, who also noted that they are equivalent to the Nahm equations, and hence are integrable.

5.2 G_2 metric on $\mathbb{R}^3$ bundle over S^4 revisited

We could attempt a similar more general construction of metrics on the $\mathbb{R}^3$ bundle over S^4. As we shall see, this does not in fact seem to be possible. It does, however, provide us with a more convenient way of writing the standard G_2 metric on this manifold.

Our starting point is the left-invariant 1-forms L_{AB} on $SO(5)$, introduced in section 2. In the earlier discussion we identified P_a and (R_1, R_2, R_3) as the 1-forms in the coset $S^7 = SO(5)/SU(2)_L$. We now divide out by a further $U(1)$ factor, associated with the 1-form R_3. The required $\mathbb{CP}^3$ principal orbits for the $\mathbb{R}^3$ bundle over S^4 are thus described by the coset

$$\mathbb{CP}^3 = \frac{SO(5)}{SU(2)_L \times U(1)_R}\,. \tag{130}$$

From (4), we can see that the following exterior derivatives lie entirely within the coset:

$$\begin{aligned} d(P_0 \wedge P_3 + P_1 \wedge P_2) &= -2R_1 \wedge (P_0 \wedge P_2 + P_3 \wedge P_1) + 2R_2 \wedge (P_0 \wedge P_1 + P_2 \wedge P_3)\,, \\ d(R_1 \wedge R_2) &= \tfrac{1}{2} R_1 \wedge (P_0 \wedge P_2 + P_3 \wedge P_1) - \tfrac{1}{2} R_2 \wedge (P_0 \wedge P_1 + P_2 \wedge P_3)\,. \end{aligned} \tag{131}$$

In particular, we see that

$$d(P_0 \wedge P_3 + P_1 \wedge P_2 + 4R_1 \wedge R_2) = 0\,. \tag{132}$$

This corresponds to the nearly-Kähler structure on $\mathbb{CP}^3$. Note, however, that there is no result lying purely within the coset if we try giving the $P_0 \wedge P_3$ and $P_1 \wedge P_2$ terms different coefficients. Thus we cannot break the S^4 base (whose coset 1-forms are P_a) apart. This is quite different from the previous example in section 5.1.

The most general metric ansatz we can consider is therefore

$$ds_7^2 = dt^2 + a^2\,(R_1^2 + R_2^2) + b^2\,P_a^2\,. \tag{133}$$

This is equivalent to the standard ansatz for the G_2 metrics on the $\mathbb{R}^3$ bundle over S^4 [1, 2].

The natural $SO(5)$-invariant ansatz for the calibrating 3-form is

$$\begin{aligned} G_{(3)} &= dt \wedge [a^2 R_1 \wedge R_2 - b^2 (P_0 \wedge P_3 + P_1 \wedge P_2)] \\ &\quad + a\,b^2\,[R_2 \wedge (P_0 \wedge P_1 + P_2 \wedge P_3) - R_1 \wedge (P_0 \wedge P_2 + P_3 \wedge P_1)]\,. \end{aligned} \tag{134}$$

From the condition $dG_{(3)} = 0$ we get

$$\frac{d(a\,b^2)}{dt} = -\tfrac{1}{2} a^2 - 2b^2\,, \tag{135}$$

while from $d{*}G_{(3)} = 0$ we get

$$2b\,\frac{db}{dt} + a = 0\,, \qquad \frac{d(a^2\,b^2)}{dt} + 4a\,b^2 = 0\,. \tag{136}$$

These imply the first-order equations $\dot a = \frac{1}{2}a^2\,b^{-2} - 2$ and $\dot b = -\frac{1}{2}a\,b^{-1}$, which are the same, after appropriate adjustment for normalisations, as those obtained in [10] for the $\mathbb{R}^3$ bundle over S^4. The solution can be written as

$$ds_7^2 = \Big(1 - \frac{\ell^4}{r^4}\Big)^{-1}\,dr^2 + r^2\,\Big(1 - \frac{\ell^4}{r^4}\Big)\,(R_1^2 + R_2^2) + \tfrac{1}{2}r^2\,P_a^2\,. \tag{137}$$

5.3 G_2 metrics for the six-function triaxial $S^3 \times S^3$ ansatz

Another class of G_2 metrics that may be studied has principal orbits that are $S^3\times S^3$. A rather general ansatz involving six radial functions was considered in [12, 13], and first-order equations for G_2 holonomy were derived. The metric for the six-function G_2 space is given by

$$ds_7^2 = dt^2 + a_i^2\,(\sigma_i - \Sigma_i)^2 + b_i^2\,(\sigma_i + \Sigma_i)^2\,, \tag{138}$$

where σ_i and Σ_i are left-invariant 1-forms for two $SU(2)$ group manifolds. It was found that for G_2 holonomy, a_i and b_i must satisfy the first-order equations

$$\begin{aligned}
\dot a_1 &= \frac{a_1^2}{4a_3\,b_2} + \frac{a_1^2}{4a_2\,b_3} - \frac{a_2}{4b_3} - \frac{a_3}{4b_2} - \frac{b_2}{4a_3} - \frac{b_3}{4a_2}\,,\\
\dot a_2 &= \frac{a_2^2}{4a_3\,b_1} + \frac{a_2^2}{4a_1\,b_3} - \frac{a_1}{4b_3} - \frac{a_3}{4b_1} - \frac{b_1}{4a_3} - \frac{b_3}{4a_1}\,,\\
\dot a_3 &= \frac{a_3^2}{4a_2\,b_1} + \frac{a_3^2}{4a_1\,b_2} - \frac{a_1}{4b_2} - \frac{a_2}{4b_1} - \frac{b_1}{4a_2} - \frac{b_2}{4a_1}\,,\\
\dot b_1 &= \frac{b_1^2}{4a_2\,a_3} - \frac{b_1^2}{4b_2\,b_3} - \frac{a_2}{4a_3} - \frac{a_3}{4a_2} + \frac{b_2}{4b_3} + \frac{b_3}{4b_2}\,,\\
\dot b_2 &= \frac{b_2^2}{4a_3\,a_1} - \frac{b_2^2}{4b_3\,b_1} - \frac{a_1}{4a_3} - \frac{a_3}{4a_1} + \frac{b_1}{4b_3} + \frac{b_3}{4b_1}\,,\\
\dot b_3 &= \frac{b_3^2}{4a_1\,a_2} - \frac{b_3^2}{4b_1\,b_2} - \frac{a_1}{4a_2} - \frac{a_2}{4a_1} + \frac{b_1}{4b_2} + \frac{b_2}{4b_1}\,,
\end{aligned} \tag{139}$$

One can look for solutions with regular Taylor expansions corresponding to a collapsing S^1, S^2 or S^3 at $t=0$. We find no such regular solutions for a collapsing S^1 or S^2, but for a collapsing S^3, we find that solutions which are regular near the associated S^3 bolt at $t=0$ have a Taylor expansion with three free parameters, and are given by

$$a_i = a_0 + \tfrac{1}{16a_0}\,t^2 + \cdots\,, \qquad b_i = -\tfrac{1}{4}t + q_i\,t^3 + \cdots\,, \tag{140}$$

where $a_0^{-2} = 64(q_1+q_2+q_3)$ (implying that $q_1+q_2+q_3>0$). A numerical analysis now shows that regularity at large distance requires that

$$q_1 \ge q_2 = q_3\,, \qquad \text{or cyclic order}\,. \tag{141}$$

Thus the only regular solutions of the six-function equations (139) are solutions also of the reduced four-function equations first obtained in [13]. Setting $a_2 = a_3$ and $b_2 = b_3$ in (139), these are

$$\begin{aligned} \dot{a}_1 &= \frac{a_1^2}{2a_2\,b_2} - \frac{a_2}{2b_2} - \frac{b_2}{2a_2}\,, \\ \dot{a}_2 &= \frac{a_2^2}{4a_1\,b_2} - \frac{a_1}{4b_2} - \frac{b_2}{4a_1} - \frac{b_1}{4a_2}\,, \\ \dot{b}_1 &= \frac{b_1^2}{4a_2^2} - \frac{b_1^2}{4b_2^2}\,, \\ \dot{b}_2 &= \frac{b_2^2}{4a_2\,a_1} - \frac{a_2}{4a_1} - \frac{a_1}{4a_2} + \frac{b_1}{4b_2}\,. \end{aligned} \tag{142}$$

Making the redefinitions

$$A = \frac{a_2^2}{a_1^2}\,, \qquad B = \frac{b_2^2}{a_1^2}\,, \qquad F = \frac{b_1\,a_2\,b_2}{a_1^3}\,, \qquad G = \frac{a_2\,b_2}{a_1}\,, \tag{143}$$

the equations become

$$\begin{aligned} A' &= 3A^2 + A(B-3) + F\,, \qquad B' = 3B^2 + B(A-3) - F\,, \\ F' &= (-4 + 3A + 3B)\,F\,, \quad G^{-1}\,G' = -2 + A + B + \frac{F}{2A} - \frac{F}{2B}\,, \end{aligned} \tag{144}$$

where a prime denotes a derivative with respect to η, which is defined by $dt = 2G\,d\eta$. Note that G is decoupled from the first three equations. If we now define

$$A = X + Y\,, \qquad B = X - Y\,, \tag{145}$$

the first three equations give

$$X' = 4X^2 - 3X + 2Y^2\,, \quad Y' = 6X\,Y - 3Y + F\,, \quad F' = 2(3X-2)\,F\,. \tag{146}$$

By calculating Y'', using the other first-order equations, and then writing $d/d\eta = X'\,d/dX$, we get the following second-order equation:

$$\begin{aligned} &(4X^2 - 3X + 2Y^2)\Big[(4X^2 - 3X + 2Y^2)\,\frac{d^2Y}{dX^2} + 4Y\Big(\frac{dY}{dX}\Big)^2 - 4(X-1)\,\frac{dY}{dX}\Big] \\ &+12(X+Y-1)\,(X-Y-1)\,Y = 0\,. \end{aligned} \tag{147}$$

Note that a special solution of this equation is

$$Y = \frac{\sqrt{3-4X}}{2}\,. \tag{148}$$

This is in fact the isolated solution that was found in [13]. It can be written as

$$\begin{aligned} a_1 &= -\tfrac{1}{2}r\,, \qquad a_2 = \tfrac{1}{4}\sqrt{3(r-\ell)(r+3\ell)}\,, \\ b_1 &= \ell\,\frac{\sqrt{r^2 - 9\ell^2}}{\sqrt{r^2-\ell^2}}\,, \qquad b_2 = -\tfrac{1}{4}\,\sqrt{3(r+\ell)(r-3\ell)}\,, \end{aligned} \tag{149}$$

where $dt = -\frac{3}{2}\ell\, dr/b_1$.

Taking the $q_2 = q_3$, our numerical analysis shows that $q_2/q_1 \geq 1$ is a non-trivial parameter characterising inequivalent solutions, which are non-singular and ALC when

$$-\tfrac{1}{2} < \frac{q_2}{q_1} < 1\,. \tag{150}$$

The limiting case $q_2/q_1 = 1$ gives an AC solution, which is in fact the previously-known G_2 metric on the spin bundle of S^3 [1, 2]. The general family in (150) includes the specific explicitly-known example (149) found in [13]. Converting to the proper-distance coordinate t, we find that the solution (149) corresponds to $q_2/q_1 = -\frac{1}{14}$.

Our numerical analysis supports the perturbative arguments given in [13], which indicated the existence of the non-trivial 1-parameter family of ALC solutions that we have found numerically. By analogy with our notation for the new ALC 8-manifolds of Spin(7) holonomy found in [3], we shall denote the explicit G_2 solution (149) of [13] by $\mathbb{B}_7$. We shall also denote the 1-parameter family of non-singular ALC solutions with $-\frac{1}{2} < q_2/q_1 < -\frac{1}{14}$ by $\mathbb{B}_7^-$, and those with $-\frac{1}{14} < q_2/q_1 < 1$ by $\mathbb{B}_7^+$. It should be noted, however, that there is no $\mathbb{A}_7$ solution of $\mathbb{R}^7$ topology, which would be analogous to the $\mathbb{A}_8$ solution on $\mathbb{R}^8$ found in [3]. This is because unlike the metrics studied in [3], where the principal orbits were spheres (S^7), which have the possibility of collapsing down smoothly to a point at the original of spherical polar coordinates, here the principal orbits are $S^3 \times S^3$, and so a smooth collapse to a point is impossible.

Note that besides the upper bound $q_2/q_1 = 1$ when we recover the original AC metric of G_2 holonomy [1, 2] on the $\mathbb{R}^4$ bundle over S^3, the lower bound, $q_2/q_1 = -\frac{1}{2}$, corresponds to the Gromov-Hausdorff limit in which we get $M_6 \times S^1$, where M_6 is the Ricci-flat Kähler metric on the deformed conifold. The various non-singular solutions are depicted in Figure 4 below.

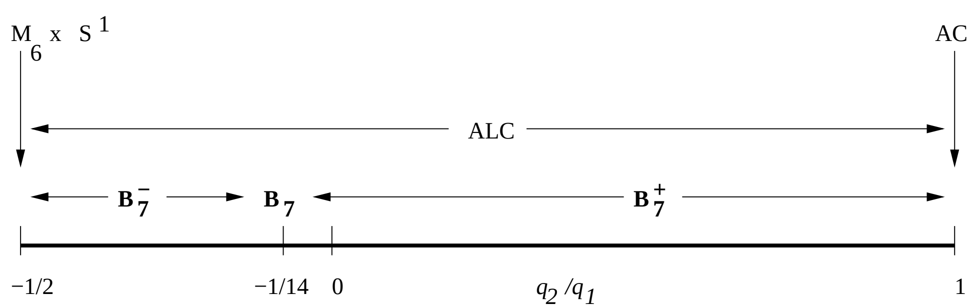

Figure 4: The non-singular G_2 metrics $\mathbb{B}_7$ and $\mathbb{B}_7^{\pm}$ as a function of q_2/q_1

6 Conclusions

In this paper, we have made a rather extensive investigation of many of the possible classes of metrics of cohomogeneity one in dimensions eight and seven that might give rise to the exceptional holonomies Spin(7) and G_2 respectively. For the case of eight dimensions, we considered first the situation where the principal orbits are topologically S^7, endowed with a homogeneous metric given by the coset $SO(5)/SO(3)$. One can view such metrics as S^3 bundles over S^4, where the S^3 fibres are themselves required to be only left-invariant under the action of $SU(2)$. The eight-dimensional metric ansatz therefore has an $SO(5)$ isometry, and involves four functions of the radial variable; three characterising homogeneous "squashings" of the S^3 fibres, and a fourth measuring the radius of the S^4 base. We obtained first-order equations for these functions, coming from the requirement of Spin(7) holonomy, and we then examined the possible solutions. We found by a numerical analysis that there should exist a family of complete and non-singular metrics with a non-trivial parameter $\lambda^2 \le 4$, which we denote by $\mathbb{C}_8$, which are topologically $\mathbb{R}^2$ bundles over $\mathbb{CP}^3$. The parameter λ characterises the degree of squashing of the minimal $\mathbb{CP}^3$ bolt, with $\lambda^2 = 4$ corresponding to the Fubini-Study metric on $\mathbb{CP}^3$. This limiting case has $SU(4)$ holonomy, and the metric has been known for a long time, but the metrics with $\lambda^2 < 4$ are new. They are ALC, and on the S^3 fibres they exhibit a similar behaviour to that seen in the Atiyah-Hitchin metric in $D = 4$.

We then considered eight-dimensional metrics of cohomogeneity one whose principal orbits are the Aloff-Wallach spaces $N(k,\ell) = SU(3)/U(1)_{k,\ell}$. We began with a more complete and explicit discussion of the Einstein metrics on $N(k,\ell)$ than has previously appeared in the literature. Earlier results showed the existence of Einstein metrics [7], gave an explicit result for an Einstein metric on $N(k,\ell)$ [8], and gave a demonstration, based on the results of [8], that there exist two inequivalent Einstein metrics on each $N(k,\ell)$ except $N(1,-1)$ [9]. In this paper, we gave an explicit construction, from first principles, of the two Einstein metrics, deriving them from the conditions for weak G_2 holonomy. These Einstein metrics form the possible bases for cones in eight-dimensional AC metrics of Spin(7) holonomy. Although we were unable to obtain the general solutions of the first-order equations for Spin(7) holonomy, we were able to find an isolated ALC solution explicitly for all (k,ℓ). In general, the metric will not be completely non-singular, but rather, will have an orbifold structure, of the local form $\mathbb{R}^4/Z_p \times \mathbb{CP}^2$, where $p = |k+\ell|$.

We also studied the solutions of the first-order equations for $N(k,\ell)$ principal orbits numerically, and in certain perturbative expansions, and found evidence for the existence of complete and non-singular metrics, both AC and ALC, for all (k, ℓ).

We then turned our attention to seven-metrics of cohomogeneity one with G_2 holonomy. We studied the first-order system for the case where the principal orbits are the flag manifold $SU(3)/(U(1)\times U(1))$. This can have three metric functions depending on the radial variable. We showed that the first-order equations implying G_2 holonomy reduce to the same ones that are encountered in one of the first-order systems for hyper-Kähler Bianchi IX metrics in $D = 4$, and hence they can be solved by the same method that was used in [11]. As in that case, it turns out that the resulting metrics are singular unless two of the metric functions are equal, in which case the system reduces to the already-studied one whose solution is the complete non-singular G_2 metric on the $\mathbb{R}^3$ bundle of self-dual 2-forms over $\mathbb{CP}^2$ [1, 2].

A second G_2 example arises if the principal orbits are $\mathbb{CP}^3$, described as an S^2 bundle over S^4. Only two metric functions are possible in this case, describing the radii of the S^2 fibres and the S^4 base, and the system reduces to the one that was solved in [1, 2], giving the non-singular metric on the $\mathbb{R}^3$ bundle of self-dual 2-forms over S^4.

A third possibility is when the principal orbits are $S^3\times S^3$, described as an S^3 bundle over S^3. In principle one can now write an ansatz with nine functions of the radial coordinate [12], although it is not clear that a first-order system of equations for G_2 holonomy can arise in this case. A simpler system with six functions (three measuring the radii of the squashed S^3 base, and three measuring the radii of the squashed S^3 fibres) was also considered in [12], and in [13], for which a first-order system implying G_2 holonomy exists. Our numerical investigations in this paper lead to the conclusion that the solutions will only be non-singular if pairs of metric functions on the base and fibre 3-spheres are set equal. This results in a four-function system, whose general solution has not been found analytically. An isolated ALC example was found in [13], and arguments for the existence of a non-trivial 1-parameter family were presented. We have analysed the system numerically in this paper, and we also find evidence for the existence of such a family of non-singular solutions.

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Note Added

After this paper was completed, the paper [51] appeared, which also studies the solutions of the first-order equations in [10] for metrics of Spin(7) holonomy with $N(k,\ell)$ principal orbits. This has an extensive overlap with our results in sections 3.2 and 3.3. In particular, [51] also obtains the explicit ALC solutions (75), and discusses the existence of more general classes of ALC metrics.

A The geometry of $SU(3)$

In order for the better understanding of the global structures of the Aloff-Wallach spaces $N(k,\ell) = SU(3)/U(1)_{k,\ell}$, it can be useful to have available an explicit parameterisation of the group $SU(3)$. One may parameterise any $SU(3)$ group element g in terms of generalised Euler angles as

$$g = U\, e^{\mathrm{i}\,\lambda_5\,\xi}\, \widetilde{U}\, e^{\frac{\mathrm{i}\sqrt{3}}{2}\,\lambda_8\,\tau}\,, \tag{151}$$

where

$$U \equiv e^{\frac{\mathrm{i}}{2}\,\lambda_3\,\phi}\, e^{\frac{\mathrm{i}}{2}\,\lambda_2\,\theta}\, e^{\frac{\mathrm{i}}{2}\,\lambda_3\,\psi}\,, \qquad \widetilde{U} \equiv e^{\frac{\mathrm{i}}{2}\,\lambda_3\,\tilde\phi}\, e^{\frac{\mathrm{i}}{2}\,\lambda_2\,\tilde\theta}\, e^{\frac{\mathrm{i}}{2}\,\lambda_3\,\widetilde{\psi}}\,, \tag{152}$$

where (θ,ϕ,ψ) are Euler angles for $SU(2)$, $(\tilde\theta,\tilde\phi,\widetilde{\psi})$ are Euler angles for another $SU(2)$, and the λ_a are the Gell-Mann matrices. The coordinate ranges are

$$0 \le \theta \le \pi\,, \qquad 0 \le \tilde\theta \le \pi\,, \qquad 0 \le \xi \le \tfrac{1}{2}\pi \tag{153}$$

for the "latitudes," while the azimuthal coordinates have the periods

$$\Delta\,\phi = 2\pi\,, \quad \Delta\,\psi = 4\pi\,, \quad \Delta\,\tilde\phi = 2\pi\,, \quad \Delta\,\widetilde{\psi} = 4\pi\,, \quad \Delta\,\tau = 2\pi\,. \tag{154}$$

Since the determination of these periods is slightly non-trivial, and errors have occurred in various published papers, we shall give an explicit derivation of the periods below.

It is useful to define left-invariant 1-forms s_i and $\tilde s_i$ for the two $SU(2)$ subgroups in the standard way:

$$U^{-1}\,dU = \tfrac{\mathrm{i}}{2}\,\lambda_i\,s_i\,, \qquad \widetilde{U}^{-1}\,d\widetilde{U} = \tfrac{\mathrm{i}}{2}\,\lambda_i\,\tilde s_i\,, \tag{155}$$

giving

$$s_1 = \cos\psi\, d\theta + \sin\psi\, \sin\theta\, d\phi\,, \quad s_2 = -\sin\psi\, d\theta + \cos\psi\, \sin\theta\, d\phi\,, \quad s_3 = d\psi + \cos\theta\, d\phi\,, \tag{156}$$

and similarly for $\tilde{s}_i$. Calculating the $SU(3)$ left-invariant 1-forms X_a defined by $g^{-1}\, dg = \frac{\rm i}{2}\lambda_a\, X_a$, and taking $\nu_1 + {\rm i}\,\nu_2 \equiv X_1 + {\rm i}\, X_2$, $\sigma_1 + {\rm i}\,\sigma_2 \equiv X_4 + {\rm i}\, X_5$, $\Sigma_1 + {\rm i}\,\Sigma_2 \equiv X_6 + {\rm i}\, X_7$, together with $\sigma_3 \equiv X_3$ and $\sigma_8 \equiv X_8$ for the two Cartan subalgebra 1-forms, we find

$$\begin{aligned}
\nu_1 + {\rm i}\,\nu_2 &= {\rm i}\,\tilde{s}_1 + \tilde{s}_2 + e^{{\rm i}\,\tilde{\psi}}\Big[{\rm i}\,\cos\xi\,[(\cos\tilde{\phi} + {\rm i}\,\sin\tilde{\phi}\,\cos\tilde{\theta})\, s_1 \\
&\qquad + (\sin\tilde{\phi} - {\rm i}\,\cos\tilde{\phi}\,\cos\tilde{\theta})\, s_2] + \tfrac14(3+\cos 2\xi)\,\sin\tilde{\theta}\, s_3\Big]\,, \\
\sigma_1 + {\rm i}\,\sigma_2 &= {\rm i}\, e^{\frac{\rm i}{2}\tilde{\psi} + \frac{3{\rm i}}{2}\tau}\Big[2e^{\frac{\rm i}{2}\tilde{\phi}}\,\cos\tfrac12\tilde{\theta}\,(d\xi - \tfrac{\rm i}{4}\,\sin 2\xi\, s_3) + e^{-\frac{\rm i}{2}\tilde{\phi}}\,\sin\xi\,\sin\tfrac12\tilde{\theta}\,(s_1 + {\rm i}\, s_2)\Big]\,, \\
\Sigma_1 + {\rm i}\,\Sigma_2 &= {\rm i}\, e^{-\frac{\rm i}{2}\tilde{\psi} + \frac{3{\rm i}}{2}\tau}\Big[2e^{\frac{\rm i}{2}\tilde{\phi}}\,\sin\tfrac12\tilde{\theta}\,(d\xi - \tfrac{\rm i}{4}\,\sin 2\xi\, s_3) - e^{-\frac{\rm i}{2}\tilde{\phi}}\,\sin\xi\,\cos\tfrac12\tilde{\theta}\,(s_1 + {\rm i}\, s_2)\Big]\,, \\
\sigma_3 &= \tilde{s}_3 + \cos\xi\,\sin\tilde{\theta}\,(\sin\tilde{\phi}\, s_1 - \cos\tilde{\phi}\, s_2) + \tfrac14(3+\cos 2\xi)\,\cos\tilde{\theta}\, s_3\,, \\
\sigma_8 &= \sqrt{3}\,(d\tau - \tfrac12\sin^2\xi\, s_3)\,.
\end{aligned} \tag{157}$$

For some purposes it is highly advantageous to introduce instead right-invariant 1-forms $\tilde{t}_i$ for the second $SU(2)$ group, defined by $d\widetilde{U}\,\widetilde{U}^{-1} = \frac{\rm i}{2}\,\lambda_i\,\tilde{t}_i$. In terms of the Euler angles $(\tilde{\theta}, \tilde{\phi}, \tilde{\psi})$, these are given by

$$\tilde{t}_1 = \cos\tilde{\phi}\, d\tilde{\theta} + \sin\tilde{\phi}\,\sin\tilde{\theta}\, d\widetilde{\psi}\,, \quad \tilde{t}_2 = \sin\tilde{\phi}\, d\tilde{\theta} - \cos\tilde{\phi}\,\sin\tilde{\theta}\, d\widetilde{\psi}\,, \quad \tilde{t}_3 = d\tilde{\phi} + \cos\tilde{\theta}\, d\widetilde{\psi}\,. \tag{158}$$

The $SU(3)$ left-invariant 1-forms ν_1, ν_2 and σ_3 then become

$$\begin{aligned}
\nu_1 + {\rm i}\,\nu_2 &= e^{{\rm i}\,\tilde{\psi}}\Big[{\rm i}\,(\cos\tilde{\phi} + {\rm i}\,\sin\tilde{\phi}\,\cos\tilde{\theta})\,[\tilde{t}_1 + \cos\xi\, s_1] \\
&\qquad + {\rm i}\,(\sin\tilde{\phi} - {\rm i}\,\cos\tilde{\phi}\,\cos\tilde{\theta})\,[\tilde{t}_2 + \cos\xi\, s_2] + \sin\tilde{\theta}\,[\tilde{t}_3 + \tfrac14(3+\cos 2\xi)\, s_3]\Big]\,, \\
\sigma_3 &= \sin\tilde{\theta}\Big[\sin\tilde{\phi}\,[\tilde{t}_1 + \cos\xi\, s_1] - \cos\tilde{\phi}\,[\tilde{t}_2 + \cos\xi\, s_2]\Big] + \cos\tilde{\theta}\,[\tilde{t}_3 + \tfrac14(3+\cos 2\xi)\, s_3]\,.
\end{aligned} \tag{159}$$

From these results, it is straightforward to show that

$$\begin{aligned}
ds_4^2 \equiv \sigma_1^2 + \sigma_2^2 + \Sigma_1^2 + \Sigma_2^2 &= 4(d\xi^2 + \tfrac14\sin^2\xi\,(s_1^2 + s_2^2) + \tfrac14\sin^2\xi\,\cos^2\xi\, s_3^2)\,, \\
ds_3^2 \equiv \nu_1^2 + \nu_2^2 + \sigma_3^2 &= (\tilde{t}_1 + \cos\xi\, s_1)^2 + (\tilde{t}_2 + \cos\xi\, s_2)^2 + [\tilde{t}_3 + \tfrac14(3+\cos 2\xi)\, s_3]^2\,.
\end{aligned} \tag{160}$$

The metric ds_4^2 is 4 times the standard Fubini-Study metric on $\mathbb{CP}^2$, and since its principal orbits at fixed ξ are $SU(2)$, this proves that ϕ and ψ must indeed have the periods given in (154). The 8-metric

$$\begin{aligned}
ds_8^2 &= \equiv \nu_1^2 + \nu_2^2 + \sigma_3^2 + \sigma_8^2 + \sigma_1^2 + \sigma_2^2 + \Sigma_1^2 + \Sigma_2^2 \\
&= (\tilde{t}_1 + \cos\xi\, s_1)^2 + (\tilde{t}_2 + \cos\xi\, s_2)^2 + [\tilde{t}_3 + \tfrac14(3+\cos 2\xi)\, s_3]^2 \\
&\qquad + 3(d\tau - \tfrac12\sin^2\xi\, s_3)^2 + 4(d\xi^2 + \tfrac14\sin^2\xi\,(s_1^2 + s_2^2) + \tfrac14\sin^2\xi\,\cos^2\xi\, s_3^2)
\end{aligned} \tag{161}$$

is then the canonical bi-invariant metric on $SU(3)$, viewed as a $U(2)$ bundle over $\mathbb{CP}^2$.

If we project the metric (161) orthogonally to $\partial/\partial\tau$, which amounts to dropping the term σ_8^2, we get a metric on the the Aloff-Wallach space $N(1,1)$, viewed as an $SO(3)$ bundle over $\mathbb{CP}^2$ (see [9]). The fact that the bundle is $SO(3)$ and not $SU(2)$ means that $\tilde{\phi}$ and $\tilde{\psi}$ must indeed have the periods given in (154). (We see from (161) that we have an $SO(3)$ bundle as opposed to $SU(2)$ since $\tilde{\phi}$ has period 2π.)

B The Atiyah-Hitchin system and the Ricci flow on $SU(2)$

The Ricci flow equations, which are encountered when studying the renomalisation group equations for the target-space metric g_{ij} in a sigma model, are

$$\frac{dg_{ij}}{d\mu} = R_{ij}\,, \tag{162}$$

where R_{ij} is the Ricci tensor of g_{ij}. For metrics of the form

$$ds^2 = A\,\sigma_1^2 + B\,\sigma_2^2 + C\,\sigma_3^2\,, \tag{163}$$

the non-vanishing components of the Ricci tensor in the triad $(\sigma_1, \sigma_2, \sigma_3)$ are

$$R_{11} = \tfrac{1}{2}A\,(A^2-(B-C)^2)\,, \quad R_{22} = \tfrac{1}{2}B\,(B^2-(C-A)^2)\,, \quad R_{33} = \tfrac{1}{2}C\,(C^2-(A-B)^2)\,. \tag{164}$$

The Ricci-flow equations are therefore

$$\frac{2}{A}\,\frac{dA}{d\mu} = A^2 - B^2 - C^2 + 2B\,C\,, \tag{165}$$

and cyclic permutations.

If we drop the terms involving b in (8), we get the Atiyah-Hitchin system

$$\begin{aligned} \dot{a}_1 &= \frac{a_1^2 - (a_2 - a_3)^2}{a_2\,a_3}\,,\\ \dot{a}_2 &= \frac{a_2^2 - (a_3 - a_1)^2}{a_3\,a_1}\,,\\ \dot{a}_3 &= \frac{a_3^2 - (a_1 - a_2)^2}{a_1\,a_2}\,. \end{aligned} \tag{166}$$

It is a curious fact that the Ricci-flow and the Atiyah-Hitchin system go into themselves under the identification $A = a_1$, $B = a_2$ and $C = a_3$, together with a suitable change of parameterisation, $dt = 2a_1\,a_2\,a_3\,d\mu$. As far as we are aware, this coincidence between first-order equations coming from a superpotential and the Ricci-flow equations occurs only in this case. It follows from the work of Atiyah and Hitchin that the Ricci-flow is completely integrable in this case, and in what follows we shall review the standard way of solving (166).

On begins by defining a new radial coordinate η by $dt = a_1\, a_2\, a_3\, d\eta$, and also introducing new variables w_i as in (13). One then has

$$\frac{d(w_1+w_2)}{d\eta} = 4w_1\, w_2\,, \qquad \frac{d(w_2+w_3)}{d\eta} = 4w_2\, w_3\,, \qquad \frac{d(w_3+w_1)}{d\eta} = 4w_3\, w_1\,. \tag{167}$$

It has been observed that there is an $SL(2,\mathbb{R})$ symmetry of this system. Namely, letting a, b, c and d be constants (nothing to do with the previous metric functions!), then if we define transformed variables v_i in place of w_i, and a transformed radial coordinate ξ in place of η, according to

$$\xi = \frac{a\,\eta+b}{c\,\eta+d}\,, \qquad w_i = -\frac{c}{2(c\,\eta+d)} + \frac{1}{(c\,\eta+d)^2}\, v_i\,, \tag{168}$$

where $a\,d - b\,c = 1$, then the equations (167) become

$$\frac{d(v_1+v_2)}{d\xi} = 4v_1\, v_2\,, \qquad \frac{d(v_2+v_3)}{d\xi} = 4v_2\, v_3\,, \qquad \frac{d(v_3+v_1)}{d\xi} = 4v_3\, v_1\,. \tag{169}$$

This allows one to transform a given solution into another, using the $SL(2\mathbb{R})$.

The equations (167) that give the Atiyah-Hitchin metric can be solved by defining a new radial coordinate r, related to η by $dr = u^2\, d\eta$, with u being a solution of

$$\frac{d^2u}{dr^2} + \tfrac14 u\, \mathrm{cosec}^{\,2} r = 0\,. \tag{170}$$

It can then be verified that the solution is

$$\begin{aligned} w_1 &= -u\, u' - \tfrac12 u^2\, \mathrm{cosec}\, r\,,\\ w_2 &= -u\, u' + \tfrac12 u^2\, \cot r\,,\\ w_3 &= -u\, u' + \tfrac12 u^2\, \mathrm{cosec}\, r\,, \end{aligned} \tag{171}$$

where u' means du/dr. The correct solution of (170) to choose for u is

$$u = \sqrt{2\sin r}\, K(\sin\tfrac12 r)\,, \tag{172}$$

where

$$K(k) \equiv \int_0^{\pi/2} \frac{d\phi}{(1-k^2\, \sin^2\phi)^{1/2}} \tag{173}$$

C Proof of positivity of $Q(A,B)$

In section 3.1.2, our proof that the conditions for Einstein metrics on $N(k,\ell)$ implied precisely the same set of solutions as the ostensibly more restrictive conditions for weak G_2 holonomy depended upon the assertion that the function $Q(A,B)$ in (60) is non-vanishing for all real positive A and B. We present a proof of this property here.

The sixth-order polynomial $Q(A,B)$ occurring in section 3.1.2 is given by

$$\begin{aligned}
Q(A,B) &= 5A^6 - 6A^5 - 5A^4 + 12A^3 - 5A^2 - 6A \\
&\quad +5B^6 - 6B^5 - 5B^4 + 12B^3 - 5B^2 - 6B \\
&\quad -A\,B\,(6A^4 - 42A^3 + 36A^2 + 36A + 6B^4 - 42B^3 + 36B^2 + 36B - 42) \\
&\quad -A^2\,B^2\,(5A^2 + 36A + 5B^2 + 36B - 12A\,B - 130) + 5\,. \qquad (174)
\end{aligned}$$

In order to show that all solutions of the Einstein equations for the seven-dimensional $N(k,\ell)$ spaces are also solutions of the weak G_2 holonomy equations, we need to establish that $Q(A,B)$ is non-vanishing whenever A and B are both positive. To do this, we define $A = x+y$, $B = x-y$, in terms of which (174) becomes

$$Q = (4x^2+1)(12x^2-12x+5)+4(16x^4+48x^3-80x^2+36x-13)\,y^2+64(4x^2-6x+3)\,y^4\,. \quad (175)$$

Note that since A and B are positive, it follows that $x > 0$, and although y can have either sign, it appears only via y^2 and y^4. Solving $Q = 0$ for y^2, we get

$$y^2 = \frac{13 - 36x + 80x^2 - 48x^3 - 16x^4 \pm (1-4x^2)\sqrt{J}}{32(3 - 6x + 4x^2)}\,, \qquad (176)$$

where $J \equiv 16x^4 + 96x^3 - 200x^2 + 120x - 71$. For y^2 to be real we must therefore have $J \ge 0$ or $x = \frac{1}{2}$. To have $J \ge 0$ (and x positive) we must have $x \ge 1.2873\ldots$. Now the coefficients of y^0 and y^4 in (175) are positive for all real x, and the coefficient of y^2 is positive for all $x > .898374\ldots$. Thus Q is positive for all x that satisfies $J \ge 0$. The case $x = \frac{1}{2}$ implies $y^2 = \frac{1}{4}$ and hence $(A,B) = (1,0)$ or $(0,1)$, both of which violate the requirement of A and B both being positive. Thus we have proved that Q is positive (and hence non-vanishing) whenever A and B are both positive. This completes the proof.

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